Rigid G-connections and nilpotency of p-curvatures

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Abstract

Motivated by Simpson's conjecture on the motivicity of rigid irreducible connections, Esnault and Groechenig demonstrated that the mod-p reductions of such connections on smooth projective varieties have nilpotent p-curvatures. In this paper, we extend their result to integrable G-connections.

Contents

1 Introduction

Let X be a smooth complex projective variety. The celebrated Riemann–Hilbert correspondence gives rise to an analytic isomorphism $\mathcal{M}_{\mathrm{dR}}(X,n) \cong \mathcal{M}_{\mathrm{B}}(X,n)$, where $\mathcal{M}_{\mathrm{dR}}(X,n)$ is the moduli space of integrable connections of rank n on X and $\mathcal{M}_{\mathrm{B}}(X,n)$ is the moduli space of complex local systems of rank n on X [Sim94b]. When X has higher dimension, the moduli spaces $\mathcal{M}_{\mathrm{dR}}(X,n)$ and $\mathcal{M}_{\mathrm{B}}(X,n)$ may have isolated points, whose associated integrable connections (or local systems) are called *rigid*. Following the nonabelian Hodge correspondence, any rigid integrable connection underlies a complex variation of Hodge structure. Furthermore, it is a complex direct factor of a $\mathbb{Q}$-variation of Hodge structure [Sim92, Theorem 5]. This leads to the notion of motivicity from a geometric aspect.

More precisely, an integrable connection (E,∇) is *motivic* (or *of geometric origin*) if there exists a dense open subvariety $U \subset X$ and a smooth projective morphism

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$f : Z \to U$ such that (E, ∇) is a direct summand of $R^i f_* \mathcal{O}_Z$ equipped with the Gauss–Manin connection.

Returning to the Riemann–Hilbert correspondence, notice that taking monodromy representations for integrable connections involves exponentiation, which is transcendental. Simpson addressed this transcendental nature in [Sim90], where he posed a question about which integrable connections defined over $\overline{\mathbb{Q}}$ have associated monodromy representations also defined over $\overline{\mathbb{Q}}$. He conjectured that such integrable connections should originate from geometry, namely the following conjecture:

Conjecture 1.1 (Simpson's standard conjecture, [Sim90])**.** *By spreading out, the $\overline{\mathbb{Q}}$-points in the intersection $\mathcal{M}_{\mathrm{B}}(X, n)(\overline{\mathbb{Q}}) \bigcap \mathcal{M}_{\mathrm{dR}}(X, n)(\overline{\mathbb{Q}})$ are motivic.*

As we have seen, each rigid integrable connection corresponds to an irreducible component of $\mathcal{M}_{\mathrm{dR}}(X, n)$. Therefore, if X is defined over $\overline{\mathbb{Q}}$, each of these zero-dimensional components is also defined over $\overline{\mathbb{Q}}$. This leads to a weaker version of the standard conjecture, stated as follows:

Conjecture 1.2 (Simpson's motivicity conjecture, [Sim92])**.** *Rigid integrable connections are motivic.*

The motivicity conjecture 1.2 is known for local systems on $\mathbb{P}^1$ with finitely many punctures by Katz [Kat96], rank 2 and rank 3 local systems on quasi-projective varieties from the work of Corlette–Simpson [CS08] and Langer–Simpson [LS18] respectively. We also refer the reader to [Esn23b, Esn23a, Sim90] for more details about the background and recent progress.

To provide general evidence for the motivicity conjecture, it is useful to examine whether rigid connections exhibit similar properties to those of motivic connections. Here, the work of Esnault and Groechenig stands out as deeply inspirational. For example, they proved that cohomologically rigid connections are integral [EG18]. Moreover, since Gauss–Manin connections in characteristic p have nilpotent p-curvatures [Kat72], Esnault and Groechenig also confirmed the nilpotency of p-curvatures of mod-p reductions of rigid connections [EG20].

Progress has also been made on analogous problems for G-connections. Specifically, Klevdal and Patrikis proved the integrality of cohomologically rigid G-connections [KP22], and Færgeman established the motivicity conjecture for G-connections on curves by demonstrating that rigid G-connections on curves are Hecke eigensheaves [Fær].

In this paper, we aim to generalize Esnault–Groechenig's result on nilpotency of p-curvatures [EG20, Theorem 1.4] to rigid integrable G-connections. Here is the main result:

Theorem 1.3. *Let X be a connected smooth projective complex variety, G a connected complex reductive group, and (E, ∇) a rigid integrable G-connection. Then there is a scheme S of finite type over $\mathbb{Z}$ over which $(X, (E, \nabla))$ has a model $(X_S, (E_S, \nabla_S))$ such that for all closed points $s \in |S|$, the restrictions (E_s, ∇_s) have nilpotent p-curvatures.*

The proof of Theorem 1.3 follows a similar strategy to that used for [EG20, Theorem 1.4]. Below is a summary of this approach for integrable G-connections.

$$
\begin{tikzcd}
\text{rigid integrable } G\text{-connections on } X \arrow[r, "(1)"] \arrow[dd, "(3)"] & \text{rigid } G\text{-Higgs bundles on } X \arrow[d, "(2)"] \\
 & \text{rigid nilpotent } G\text{-Higgs bundles on } X \arrow[d, "(3)"] \\
\text{rigid integrable } G\text{-connections on } X_s & \text{rigid nilpotent } G\text{-Higgs bundles on } X_s \arrow[l, "(4)"']
\end{tikzcd}
$$

(1) By Corlette–Simpson correspondence (Theorem 2.9), rigid integrable G-connections correspond to rigid G-Higgs bundles.

(2) Rigid G-Higgs bundles are actually nilpotent (Lemma 4.7).

(3) The approach to find such an arithmetic model is the same as [EG20, Lemma 3.1 and Proposition 3.3].

(4) The nonabelian Hodge correspondence in positive characteristic for principal bundles (Theorem 2.11) preserves rigidity (Lemma 4.9), and thus the corresponding rigid integrable G-connections on X_s have nilpotent p-curvatures.

Compared with Esnault and Groechenig's proof, we need a nonabelian Hodge correspondence in positive characteristic for principal bundles (Step (4)) and a Hitchin morphism for G-Higgs bundles (Step (2)).

First, the nonabelian Hodge correspondence in positive characteristic, known as the Ogus—Vologodsky correspondence, was established by Ogus and Vologodsky [OV07] and further studied by groups of people [CZ15, LSZ19]. A key question was how to extend this correspondence to principal bundles. Recently, Sheng, Wang, and the third author achieved this generalization for principal bundles using the exponential twisting approach of Lan–Sheng–Zuo [LSZ15] (see [SSW, Theorems 3.1 and 3.10] or Theorem 2.11).

Second, the Hitchin morphism for Higgs bundles on higher dimensional varieties has been well-known for decades (see [Sim94b, §6 Hitchin's proper map] for instance). For G-Higgs bundles, Chen–Ngô recently proposed a framework to study this morphism [CN20], called the spectral data morphism. With the help of the spectral data morphism, Step (2) follows directly, i.e., rigid G-Higgs bundle is nilpotent. We also use spectral data morphism to prove that the nonabelian Hodge correspondence preserves rigidity (Lemma 4.9).

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2 Recollections on nonabelian Hodge correspondence

2.1 G-Higgs bundles and flat G-bundles

Since we consider both G-Higgs bundles and flat G-bundles in mixed characteristic, the following definitions of G-Higgs bundles and flat G-bundles are given over a perfect field k. Let X be a smooth projective variety over k. In the case of positive characteristic, we have the following diagram

$$\begin{tikzcd} X \arrow[r, "F"] \arrow[rr, bend left, "F_X"] \arrow[dr] & X' \arrow[r, "\pi_{X/k}"] \arrow[d] & X \arrow[d] \\ & \operatorname{Spec} k \arrow[r, "F_k"] & \operatorname{Spec} k \end{tikzcd}$$

where

- F_X and F_k are the absolute Frobenius morphisms;
- X' is the pullback of X along F_k;
- $F : X \to X'$ is the relative Frobenius morphism.

Let Ω_X denote the cotangent sheaf and G be a connected split reductive group G over k with Lie algebra $\mathfrak{g}$. Given a G-bundle V on X, we denote by $V(\mathfrak{g}) := V \times_G \mathfrak{g}$ the adjoint bundle.

2.1.1 G-Higgs bundles

Definition 2.1. A *G-Higgs bundle* on X is a pair (V, θ), where V is a G-bundle and $\theta \in H^0(X, V(\mathfrak{g}) \otimes \Omega_X)$ is a section satisfying the integrability condition $\theta \wedge \theta = 0$. Such a section θ is called a *Higgs field*.

Let (V, θ) be a G-Higgs bundle. Taking a parabolic subgroup $P \subseteq G$ and a reduction of structure group $\sigma : X \to V/P$, we define the product V_σ via the following diagram

$$\begin{tikzcd} V_\sigma \arrow[r, dotted] \arrow[d, dotted] & V \arrow[d] \\ X \arrow[r, "\sigma"] & V/P\,. \end{tikzcd}$$

Clearly, V_σ is a P-bundle on X. A reduction of structure group σ is *compatible* with the Higgs field θ if there is a lifting

$$\begin{tikzcd} & V_\sigma(\mathfrak{p}) \otimes \Omega_X \arrow[d] \\ X \arrow[ur, dotted, "\theta_\sigma"] \arrow[r, "\theta"'] & V(\mathfrak{g}) \otimes \Omega_X\,, \end{tikzcd}$$

where $V_\sigma(\mathfrak{p})$ is the adjoint bundle of V_σ. Taking a character $\chi : P \to \mathbb{G}_m$, we obtain a $\mathbb{G}_m$-bundle $\chi_*(V_\sigma)$ and thus a line bundle on X. Now, we will give the R-stability condition on G-Higgs bundles. We refer the reader to [Ram75, Ram96a, Ram96b] for Ramanathan's original definition for principal bundles and [KSZ24] for parahoric objects.

Definition 2.2. A G-Higgs bundle (V, θ) is *R-semistable* (resp. *R-stable*) if

- for any proper parabolic subgroup $P \subseteq G$;
- for any θ-compatible reduction of structure group $X \to V/P$;
- for any anti-dominant character $\chi : P \to \mathbb{G}_m$ acting trivially on the center of P,

we have

$$\deg \chi_*(V_\sigma) \geq 0 \,(\text{resp. } >)\,.$$

2.1.2 Flat G-bundles

Now we move to flat G-bundles and refer the reader to [CZ15, Appendix] for more details.

Definition 2.3. Let E be a G-bundle. An *integrable G-connection* ∇ on E is an integrable connection $\nabla : \mathcal{O}_E \to \mathcal{O}_E \otimes \Omega_X$ compatible with the G-action on E. Equivalently, an *integrable G-connection* is a section $T_X \to \mathrm{At}(E)$ of Lie algebroids of the following short exact sequence

$$0 \to E(\mathfrak{g}) \to \mathrm{At}(E) \to T_X \to 0,$$

where $\mathrm{At}(E)$ is the Atiyah Lie algebroid of E. A *flat G-bundle* is a pair (E, ∇), where E is a G-bundle and ∇ is an integrable G-connection. Sometimes, such a pair (E, ∇) is called an *integrable G-connection* for convenience. Denote by

$$\psi := \psi_\nabla : F^* T_{X'} \to E(\mathfrak{g})$$

the *p-curvature* associated to ∇, which is adjoint to the map

$$T_X \to E(\mathfrak{g}), \quad v \mapsto \nabla(v)^p - \nabla(v^{[p]}).$$

Let (E, ∇) be an integrable G-connection. Given a reduction of structure group $\varsigma : X \to E/P$, we have a map $E_\varsigma \to E$. The reduction of structure group ς is ∇-*compatible* if the connection ∇ induces an integrable connection $\nabla_\varsigma : \mathcal{O}_{V_\varsigma} \to \mathcal{O}_{V_\varsigma} \otimes \Omega_X$ such that the diagram commutes

$$\begin{tikzcd} \mathcal{O}_V \arrow[r, "\nabla"] \arrow[d] & \mathcal{O}_V \otimes \Omega_X \arrow[d] \\ \mathcal{O}_{V_\varsigma} \arrow[r, dotted, "\nabla_\varsigma"] & \mathcal{O}_{V_\varsigma} \otimes \Omega_X . \end{tikzcd}$$

Definition 2.4. An integrable G-connection (E, ∇) is *R-semistable* (resp. *R-stable*), if

- for any proper parabolic subgroup $P \subsetneq G$;
- for any ∇-compatible reduction of structure group $\varsigma : X \to E/P$;
- for any antidominant character $\chi : P \to \mathbb{G}_m$ acting trivially on the center of P,

we have

$$\deg \chi_*(V_\varsigma) \geq 0 \text{ (resp. } > 0).$$

2.1.3 Nilpotency

We start with the case of characteristic zero. Let k be a field of characteristic zero and let R be a finitely generated k-algebra. We introduce the following definition of nilpotency.

Definition 2.5. An element $x \in \mathfrak{g}(R)$ is *nilpotent of exponent* $\leq n-1$ if it lies in a nilpotent Lie subalgebra of nilpotency class $\leq n-1$. A collection of elements $\{x_i\}_{i \in I} \subseteq \mathfrak{g}(R)$, where I is an index set, is *nilpotent of exponent* $\leq n-1$ if they are included in a nilpotent Lie subalgebra of nilpotency class $\leq n-1$.

If $x \in \mathfrak{g}(R)$ is nilpotent, it corresponds to an element in $G(R)$ via the exponential map. We briefly review this well-known fact and refer the reader to [DG70, Chapitre II. and Chapitre IV.] for more details and relevant properties of the exponential map. There is an exact sequence

$$0 \to \mathfrak{g}(R) \to G(R[\varepsilon]/(\varepsilon^2)) \to G(R) \to 0.$$

For each element $x \in \mathfrak{g}(R)$, there exists a unique element in $G(R[[T]])$ induced by the morphism $\mathfrak{g}(R) \to G(R[\varepsilon]/(\varepsilon^2))$ and denote it by $\exp(Tx)$. If x is nilpotent, we have $\exp(Tx) \in G(R[T])$, and thus we obtain an element in $G(R)$ by taking $T = 1$ and denote it by $\exp(x)$. Moreover, the above discussion induces an isomorphism $\mathfrak{u} \to U$ via Cambell–Hausdorff series, where U is a unipotent algebraic group and $\mathfrak{u}$ is its Lie algebra. This morphism is denoted by $\exp : \mathfrak{u} \to U$ and is called the exponential map.

Now we introduce the definition of nilpotent G-Higgs bundles in the relative case, and when $R = k$, it coincides with [SSW, Definition 2.10].

Definition 2.6. Let $S = \operatorname{Spec} R$ be a k-scheme, where R is a finitely generated k-algebra. An S-family of G-Higgs bundles (V, θ) on X is *nilpotent of exponent* $\leq n-1$ if there exists a covering of X by open affine subsets U such that the set $\{\theta|_{U_S}(\partial) \mid \partial \text{ is a section of } T_{U_S/S}\} \subseteq \mathfrak{g}(\mathcal{O}_{U_S})$, where $U_S := U \times_k S$, is nilpotent of exponent $\leq n-1$.

For positive characteristics, the nilpotent elements considered in this paper are always assumed to be obtained by a mod p reduction. Here is a precise description. Let k be a perfect field in characteristic p. There is a natural morphism $\mathbb{Z}_{(p)} \to \mathbb{F}_p \to k$. Following the argument in [Ser94, §4] and [Sei00, §5], there is an isomorphism $\overline{\exp} : \mathfrak{u} \to U$, where U is a unipotent algebraic group over k of nilpotency class $\leq p-1$ and $\mathfrak{u}$ is the Lie algebra of U, induced by the exponential map in characteristic zero via the base change $\mathbb{Z}_{(p)} \to k$. An element $x \in \mathfrak{g}$ is *nilpotent of exponent* $\leq p-1$, if it is included in a nilpotent Lie subalgebra of nilpotency class $\leq p-1$ obtained in this way. Now we give the definition of nilpotent G-Higgs bundles and nilpotent integrable G-connections in positive characteristics.

Definition 2.7. Let S be a k-scheme. An S-family of G-Higgs bundles (V, θ) (resp. integrable G-connections (E, ∇)) on X is *nilpotent of exponent* $\leq p-1$ if there exists a covering of X by open affine subsets U such that the set $\{\theta|_{U_S}(\partial) \mid \partial \text{ is a section of } T_{U_S/S}\}$ (resp. $\{\psi|_{U_S}(\partial) \mid \partial \text{ is a section of } F^*T_{U'_S/S}\}$) is nilpotent of exponent $\leq p-1$. Moreover, such G-Higgs bundles (resp. integrable G-connections) are also called $[p]$-nilpotent.

2.2 Corlette–Simpson correspondence

In this subsection, we briefly overview the Corlette–Simpson correspondence, also called the nonabelian Hodge correspondence. Let X be a smooth projective variety over $\mathbb{C}$. The Corlette–Simpson correspondence [Sim94a, Sim94b] on X is given as a one-to-one correspondence among three objects:

- (poly)stable Higgs bundles on X with vanishing Chern classes;
- (semi)simple flat bundles on X;
- (semi)simple GL-representations of the fundamental group $\pi_1(X)$.

Simple flat bundles (or integrable connections) are also called irreducible flat bundles (or integrable connections). Moreover, this correspondence induces homeomorphisms among three moduli spaces

$$\mathcal{M}_{\mathrm{Dol}}(X, n) \cong \mathcal{M}_{\mathrm{dR}}(X, n) \cong \mathcal{M}_{\mathrm{B}}(X, n),$$

where

- $\mathcal{M}_{\mathrm{Dol}}(X, n)$ is the moduli space of semistable Higgs bundles of rank n on X with vanishing Chern classes, which is called the *Dolbeault moduli space*;

- $\mathcal{M}_{\mathrm{dR}}(X,n)$ is the moduli space of semisimple flat bundles of rank n on X, which is called the *de Rham moduli space*;

- $\mathcal{M}_{\mathrm{B}}(X,n)$ is the moduli space of semisimple GL_n-representations of the fundamental group, which is called the *Betti moduli space*.

Remark 2.8. We want to comment on Higgs bundles in the Corlette–Simpson correspondence. For the Dolbeault side of the correspondence, imposing the condition that all Chern classes vanish is crucial. This condition leads to the following two important properties:

- Firstly, semistable Higgs sheaves are exactly Higgs bundles [Sim94b, Proposition 6.6].

- Secondly, the Gieseker stability condition, also called the P-stability condition, is equivalent to the slope stability condition [Sim94b, Remark below Corollary 6.7].

In conclusion, it is enough to consider (poly)stable Higgs bundles under the slope stability condition in the Corlette–Simpson correspondence.

Now, we come to principal objects. Let G be a connected complex reductive group. We recall the following definitions from [Sim92, §6, Principal objects]. A G-Higgs bundle is *of vanishing Chern classes* if the adjoint Higgs bundle is of vanishing Chern classes. Moreover, it implies that the associated Higgs bundle is of vanishing Chern classes for any faithful representation. A G-Higgs bundle is *(semi)stable* if there exists a faithful representation of G such that the associated Higgs bundle is P-(semi)stable. Then, the Corlette–Simpson correspondence for principal objects is given as follows:

Theorem 2.9 ([Sim94b, Theorem 9.11 & Lemma 9.14])**.** *There is a homeomorphism between $\mathcal{M}_{\mathrm{Dol}}(X,G)$ and $\mathcal{M}_{\mathrm{dR}}(X,G)$ and there is a complex analytic isomorphism between $\mathcal{M}_{\mathrm{dR}}(X,G)$ and $\mathcal{M}_{\mathrm{B}}(X,G)$, where*

- *$\mathcal{M}_{\mathrm{Dol}}(X,G)$ is the moduli space of semistable G-Higgs bundles on X with vanishing Chern classes;*

- *$\mathcal{M}_{\mathrm{dR}}(X,G)$ is the moduli space of semisimple integrable G-connections on X;*

- *$\mathcal{M}_{\mathrm{B}}(X,G)$ is the moduli space of semisimple G-representations of the fundamental group.*

Remark 2.10. In the case of curves, the R-stability condition of G-bundles V can be interpreted as the slope stability condition for the associated bundles under a faithful representation $G \to \mathrm{GL}(W)$ (see [Ram75, Lemma 3.3] and [Ram96a, Proposition 3.17] for instance). On higher dimensional varieties, the R-stability condition of G-bundles is given by the reduction of structure group $U \to (V|_U)/P$ on an open subset U such that $\mathrm{codim}(X\backslash U) \geq 2$ [AB01, Definition 1.1]. In such higher dimensional cases, a G-Higgs bundle is R-semistable if and only if its adjoint Higgs bundle is semistable [AB01, Lemma 4.7]. For a discussion in mixed characteristics, see [GLSS08, §3.2].

In the context of the Corlette–Simpson correspondence, the Higgs bundles are always of vanishing Chern classes, which implies that all Higgs subsheaves are indeed subbundles, as discussed in Remark 2.8. Therefore, in the G-version of the correspondence, it suffices to consider reductions of structure group $X \to V/P$. This is also the reason why we give the stability condition (Definition 2.2 and 2.4) as above.

In conclusion, the Dolbeault moduli space $\mathcal{M}_{\mathrm{Dol}}(X, G)$ of principal objects can be regarded as the moduli space of R-semistable G-Higgs bundles on X with vanishing Chern classes.

2.3 Ogus–Vologodsky correspondence for principal objects

We review the nonabelian Hodge correspondence for principal objects in positive characteristics. This correspondence is an analog of Ogus–Vologodsky correspondence via the approach of exponential twisting introduced in [LSZ15]. We refer the reader to [SSW, §2 & §3] for more details. In this subsection, X is a smooth projective variety over a perfect field k in positive characteristic p, $S = \operatorname{Spec} R$, where R is a finitely generated k-algebra, and G is a split connected reductive group over k.

Theorem 2.11 ([SSW, Theorem 3.1 & Theorem 3.10])**.** *Suppose that X is $W_2(k)$-liftable. The category of S-family of nilpotent G-Higgs bundles on X of exponent $\leq p-1$ and the category of S-family of nilpotent integrable G-connections on X of exponent $\leq p-1$ are equivalent. Moreover, the equivalence preserves the R-stability condition.*

Remark 2.12. Although [SSW, Theorem 3.1] is only stated for $S = \operatorname{Spec} k$, we briefly explain in this remark why the same proof holds in the relative case. There are three main techniques used in the proof of [SSW, Theorem 3.1]: Cartier descent [Kat70, Theorem 5.1], the Deligne–Illusie lemma [DI87, §2] and the exponential map. The first two are given in the relative case in the literature. For the third one, as explained in Section 2.1.3, a relative version of the exponential map in characteristic zero is detailed in [DG70]. The exponential map $\overline{\exp}$ in positive characteristic is induced from the characteristic zero case via the base change $\mathbb{Z}_{(p)} \to k$ under the assumption that the exponent of the nilpotency is less than p. Therefore, there exists a relative version of the exponential map in positive characteristics.

Remark 2.13. In this paper, we are mostly interested in the case where the characteristic p is large, as in such cases nilpotent integrable G-connections or nilpotent G-Higgs bundles are automatically nilpotent of exponent $\leq p-1$.

When $S = \operatorname{Spec} k$, a lower bound of the characteristic is given by the Coxeter number. A brief explanation follows: Let h be the Coxeter number of G. Suppose that $p > h$. Then any nilpotent element $a \in \mathfrak{g}$ is $[p]$-nilpotent (see [McN21, Proposition 4.2.4] for instance). Therefore, any nilpotent integrable G-connections or nilpotent G-Higgs bundles on X are nilpotent of exponent $\leq p-1$.

Another situation concerned in this paper is that R is an Artin k-algebra, for example, $R = k[\varepsilon]/(\varepsilon^D)$, where D is a positive integer. In this case, we can find a large number p such that any nilpotent element $x \in \mathfrak{g}(R)$ is $[p]$-nilpotent. In fact, the existence of such p

can be reduced to the case of $\mathfrak{sl}_n(R)$ by fixing a faithful representation $G \hookrightarrow \mathrm{SL}_n$ because the nilpotency class of a unipotent subgroup in G is the same as its image in SL_n, and moreover, the case of $\mathfrak{sl}_n$ has been shown in [EG20, Proof of Proposition 3.5].

Sketch of the proof of Theorem 2.11. Let S be a k-scheme. We consider the relative case $X_S \to S$ and define the following categories:

- $\mathrm{HIG}_{p-1}(X'_S/S, G)$: the category of S-families of nilpotent G-Higgs bundles on X of exponent $\leq p-1$;
- $\mathrm{MIC}_{p-1}(X_S/S, G)$: the category of S-families of nilpotent integrable G-connections on X of exponent $\leq p-1$.

We remind the reader that we use the same notations for the corresponding stacks in the following sections. As mentioned in Remark 2.12, the theorem is proven by establishing two functors:

$$C_{\exp}^{-1} : \mathrm{HIG}_{p-1}(X'_S/S, G) \to \mathrm{MIC}_{p-1}(X_S/S, G),$$
$$C_{\exp} : \mathrm{MIC}_{p-1}(X_S/S, G) \to \mathrm{HIG}_{p-1}(X'_S/S, G).$$

Here, we only outline the construction of the inverse Cartier transform $C_{\exp}^{-1}$. In the proof, we need a lifting assumption and we refer the reader to [DI87, Remarques 2.2] and [OV07, §1.1] for more details.

We fix an affine open covering $\{U_i\}_{i\in I}$ of X, where I is the index set. Deligne and Illusie [DI87, §2] constructed the following elements:

$$\zeta_i : F^*\Omega_{U'_i/S} \to \Omega_{U_i/S} \quad \text{and} \quad h_{ij} : F^*\Omega_{U'_{ij}/S} \to \mathcal{O}_{U_{ij}},$$

such that

$$h_{ij} + h_{jk} = h_{ik} \quad \text{and} \quad \zeta_j - \zeta_i = dh_{ij}.$$

Let (V, θ) be an object in $\mathrm{HIG}_{p-1}(X'_S/S, G)$. Define $(V_i, \theta_i) := (V|_{(U_i)_S}, \theta|_{(U_i)_S})$. By Cartier descent, let $\nabla_{can,i}$ denote the canonical G-connection on $E_i := F^*V_i$. We then define a new integrable G-connection on E_i

$$\nabla_i := \nabla_{can,i} + \zeta_i(F^*\theta_i),$$

and glue $(E_i, \nabla_i)_{i\in I}$ via $G_{ij} := \overline{\exp}(h_{ij}(F^*\theta_i))$, where the existence the exponential map is explained in §2.1.3. Therefore, we obtain an integrable G-connection (E, ∇) on X_S. □

3 Spectral data morphism

In this section, we go back to the setup in mixed characteristics that X is a smooth projective variety of dimension d over a perfect field k. Denote by $\mathfrak{g}$ the Lie algebra of G. Define

$$\mathfrak{C}^d_{\mathfrak{g}} := \{(\theta_1, \ldots, \theta_d) \,|\, [\theta_i, \theta_j] = 0, 1 \leq i, j \leq d\} \subseteq \mathfrak{g}^d,$$

which is called the *commuting scheme*. There is a natural GL_d-action on $\mathfrak{C}_{\mathfrak{g}}^d$. We consider the stack $[\mathfrak{C}_{\mathfrak{g}}^d/(G\times \mathrm{GL}_d)]$. Let $X\to [*/\mathrm{GL}_d]$ be the morphism corresponding to the cotangent bundle T_X^*. Since there is a natural morphism $[\mathfrak{C}_{\mathfrak{g}}^d/(G\times \mathrm{GL}_d)]\to [*/\mathrm{GL}_d]$, the moduli stack $\mathrm{HIG}(X,G)$ of G-Higgs bundles on X is defined as sections of the fiber product $X\times_{[*/\mathrm{GL}_d]}[\mathfrak{C}_{\mathfrak{g}}^d/(G\times \mathrm{GL}_d)]$, i.e.,

$$\mathrm{HIG}(X,G):=\mathrm{Sect}(X, X\times_{[*/\mathrm{GL}_d]}[\mathfrak{C}_{\mathfrak{g}}^d/(G\times \mathrm{GL}_d)]).$$

The natural map $[\mathfrak{C}_{\mathfrak{g}}^d/G]\to \mathfrak{C}_{\mathfrak{g}}^d/\!\!/G$ induces a morphism of stacks

$$\mathrm{sd}_{X,G}:\mathrm{HIG}(X,G)\to \mathscr{B}(X,G),$$

where

$$\mathscr{B}(X,G):=\mathrm{Sect}(X,X\times_{[*/\mathrm{GL}_d]}[(\mathfrak{C}_{\mathfrak{g}}^d/\!\!/G)/\mathrm{GL}_d]).$$

In this paper, the map $\mathrm{sd}_{X,G}$ is called *the spectral data morphism*, and if there is no ambiguity, we use the notation $\mathrm{sd}:=\mathrm{sd}_{X,G}$ for simplicity. Also, the spectral data morphism induces one

$$\mathcal{M}_{\mathrm{Dol}}(X,G)\to \mathscr{B}(X,G)$$

for the Dolbeault moduli space because the objects in the Dolbeault moduli space are G-Higgs bundles by Remarks 2.8 and 2.10, and we still use the same notation sd for this morphism. Moreover, when $G=\mathrm{GL}_n$ or SL_n, $\mathscr{B}(X,G)$ is a closed subscheme of the Hitchin base $\mathscr{A}(X,G)$ [CN20].

Remark 3.1. The space of spectral data, introduced by Chen–Ngô, is defined as

$$\mathrm{Sect}(X,X\times_{[*/\mathrm{GL}_d]}[(\mathfrak{t}^d/\!\!/W)/\mathrm{GL}_d]).$$

They conjectured the following isomorphism [CN20, Conjecture 3.1]

$$\mathfrak{C}_{\mathfrak{g}}^d/\!\!/G\cong \mathfrak{t}^d/\!\!/W\,, \tag{$\bullet$}$$

where $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{g}$ and W is the Weyl group. This isomorphism can be regarded as a higher dimensional Chevalley restriction theorem. If this conjecture holds, the space of spectral data we consider is exactly the one given by Chen–Ngô.

Example 3.2. When $G=\mathrm{GL}_n$, the base of spectral data $\mathscr{B}(X,G)$ is isomorphic to the scheme of sections of relative Chow variety, i.e.

$$\mathscr{B}(X,\mathrm{GL}_n)\cong \mathrm{Sect}(X,\mathrm{Chow}_n(T_X^*/X)),$$

where $\mathrm{Chow}_n(T_X^*/X)\cong \overbrace{T_X^*\times_X\cdots\times_X T_X^*}^{n}/\mathfrak{S}_n$ and $\mathfrak{S}_n$ is the symmetric group.

Now we collect some related properties of the spectral data morphism used in this paper, as follows.

Lemma 3.3. *Suppose that the characteristic p is positive. Let* $\mathrm{MIC}(X, G)$ *be the stack of integrable G-connections on X (or stack of G-local systems on X). There is a morphism*

$$\mathrm{sd}_p : \mathrm{MIC}(X, G) \to \mathscr{B}(X', G)$$

induced by the p-curvature, which is called the p-spectral data morphism. Moreover, restricting to the nilpotent case (of exponent $\leq p-1$), we have the following commutative diagram

$$\begin{tikzcd} \mathrm{HIG}_{p-1}(X', G) \arrow[rr, dotted, bend left, "C_{\exp}^{-1}"] \arrow[dr, "\mathrm{sd}"'] & & \mathrm{MIC}_{p-1}(X, G) \arrow[ll, dotted, bend left, "C_{\exp}"] \arrow[dl, "\mathrm{sd}_p"] \\ & \mathscr{B}(X', G) & \end{tikzcd}.$$

More precisely, the inverse Cartier and Cartier transform given in Theorem 2.11 preserve spectral data morphisms.

Proof. The proof of the existence of the p-spectral data morphism is the same as that of [CZ15, Proposition 3.1]. The second statement follows directly from the proof (or construction) of Theorem 2.11 (see *Sketch of the proof* of Theorem 2.11 for instance). Moreover, in the case of curves, this is a particular case of Chen–Zhu's result [CZ15, Theorem 1.2] by restricting to the nilpotent case. $\square$

4 Nilpotency of p-curvatures

4.1 Rigidity and existence of nice models

In this subsection, we go back to the setup in §2.2 about the Corlette–Simpson correspondence and let X be a smooth projective variety over $\mathbb{C}$.

Definition 4.1. For a scheme $\mathcal{X} \to S$, we denote by $\mathcal{X}^{\mathrm{rig}}$ the maximal open subscheme such that $\mathcal{X}^{\mathrm{rig}} \to S$ is quasi-finite at all points of $\mathcal{X}^{\mathrm{rig}}$.

Definition 4.2. A R-stable G-Higgs bundle (resp. R-stable integrable G-connection) is called *rigid* if every nearby G-Higgs bundle (resp. integrable G-connection) can be deformed to it, or equivalently, if the corresponding point in the Dolbeault moduli space $\mathcal{M}_{\mathrm{Dol}}(X_S/S, G)$ (resp. de Rham moduli space $\mathcal{M}_{\mathrm{dR}}(X_S/S, G)$) is isolated. Denote by $\mathcal{M}^{\mathrm{rig}}_{\mathrm{Dol}}(X_S/S, G) \subseteq \mathcal{M}_{\mathrm{Dol}}(X_S/S, G)$ (resp. $\mathcal{M}^{\mathrm{rig}}_{\mathrm{dR}}(X_S/S, G) \subseteq \mathcal{M}_{\mathrm{dR}}(X_S/S, G)$) the open subscheme of rigid G-Higgs bundles (resp. integrable G-connections).

Based on the above definition, here is an equivalent description of rigidity from deformation theory [EG20, proof of Lemma 3.4], which does not depend on the existence of the moduli space and work in mixed characteristics. We only give the statement for G-Higgs bundles.

Lemma 4.3. *A G-Higgs bundle (V, θ) is not rigid if and only if there exists a positive dimension geometrically irreducible k-scheme C of finite type, a C-family of G-Higgs bundles (V_C, θ_C) and two closed points $c_0, c_1 \in C(k')$ defined over a finite field extension k'/k such that $(V_C, \theta_C)_{c_0} \cong (V, \theta)_{k'}$ and $(V_C, \theta_C)_{c_0}$ and $(V_C, \theta_C)_{c_1}$ are not isomorphic over the algebraic closure $\bar{k}$.*

The following definition is a lemma given in [EG20, Lemma 3.1].

Definition 4.4. A scheme S is called an *arithmetic model* if it satisfies the following conditions

(1) S is of finite type and smooth over $\operatorname{Spec} \mathbb{Z}$;

(2) S has a unique generic point η and there is an embedding of fields $k(\eta) \subseteq \mathbb{C}$;

(3) we have $\operatorname{Spec} \mathbb{C} \times_S X_S \cong X$, where the product is taken along the embedding given in (2).

Proposition 4.5 ([EG20, Proposition 3.3])**.** *There exists an arithmetic model S such that*

- *Each rigid integrable G-connection (E, ∇) can be spread out to (E_S, ∇_S) on X_S, which is R-stable over geometric points.*
- *For every rigid G-Higgs bundle (V, θ), there exists a spreading-out (V_S, θ_S) which is R-stable over geometric points and θ_S is nilpotent.*
- *(E_S, ∇_S) (resp. (V_S, θ_S)) induces a section*

$$[E_S, \nabla_S] : S \to \mathcal{M}_{\mathrm{dR}}(X_S/S, G) \quad (\textit{resp. } [V_S, \theta_S] : S \to \mathcal{M}_{\mathrm{Dol}}(X_S/S, G))$$

factoring through $\mathcal{M}^{\mathrm{rig}}_{\mathrm{dR}}(X_S/S, G)$ (resp. $\mathcal{M}^{\mathrm{rig}}_{\mathrm{Dol}}(X_S/S, G)$).

- *For each $y \in |\mathcal{M}^{\mathrm{rig}}_{\mathrm{dR}}(X_S/S, G)|$ (resp. $y \in |\mathcal{M}^{\mathrm{rig}}_{\mathrm{Dol}}(X_S/S, G)|$), there exists (E_S, ∇_S) (resp. (V_S, θ_S)) such that y belongs to the set-theoretic image $[E_S, \nabla_S](|S|)$ (resp. $[V_S, \theta_S](|S|)$).*

Remark 4.6. In [EG20, §3], the authors fix a line bundle L throughout the paper, treated as the fixed determinant line bundle. Therefore, the case they consider is actually for rigid (twisted) integrable SL_n-connections.

4.2 Proof of the main result

In this section, X is always a connected smooth projective variety over $\mathbb{C}$, and we use the notation Z for a smooth projective variety over a perfect field k in mixed characteristics.

Lemma 4.7. *Rigid G-Higgs bundles over Z have nilpotent Higgs field.*

Proof. As vector spaces, scalar multiplication gives a $\mathbb{G}_m$-action on $\mathfrak{g}^d$. This $\mathbb{G}_m$-action on $\mathfrak{g}^d$ gives one on the commuting scheme $\mathfrak{C}^d_{\mathfrak{g}}$, and thus induces a $\mathbb{G}_m$-action on $\mathfrak{C}^d_{\mathfrak{g}}/\!\!/G$. Then the above $\mathbb{G}_m$-actions on $\mathfrak{C}^d_{\mathfrak{g}}$ and $\mathfrak{C}^d_{\mathfrak{g}}/\!\!/G$ induce $\mathbb{G}_m$-actions on the moduli stacks $\mathrm{HIG}(X,G)$ and $\mathscr{B}(X,G)$ respectively, which are compatible with the spectral data morphism $\mathrm{sd}:\mathrm{HIG}(X,G)\to\mathscr{B}(X,G)$.

With respect to the $\mathbb{G}_m$-actions given above, the proof of this lemma is an analog of [EG20, Lemma 2.1]. We include it here for completeness. Let (V,θ) be a rigid G-Higgs bundle on Z. If θ is not nilpotent, then image $\mathrm{sd}((V,\theta))$ will be nontrivial. Since $\mathbb{G}_m$-action is compatible with the spectral data morphism, we obtain a natural $\mathbb{G}_m$-family $(V,\lambda\theta)$, which is a nontrivial deformation of (V,θ). By Lemma 4.3, this is a contradiction. □

Lemma 4.8. *Suppose that the characteristic p is pretty good and big. Then the inverse Cartier transform $C^{-1}_{\exp}$ given in Theorem 2.11 sends a rigid G-Higgs bundle on Z' to an integrable G-connection on Z with nilpotent p-curvature.*

Proof. By Lemma 4.7, a rigid G-Higgs bundle (V,θ) on Z' is nilpotent. Therefore, the integrable G-connection $C^{-1}_{\exp}(V,\theta)$ has nilpotent p-curvature due to the commutativity of the following diagram given in Section 3.

$$\begin{tikzcd}
\mathrm{HIG}_{p-1}(Z',G) \arrow[rr, "C^{-1}_{\exp}"] \arrow[dr, "\mathrm{sd}"'] & & \mathrm{MIC}_{p-1}(Z,G) \arrow[dl, "\mathrm{sd}_p"] \\
& \mathscr{B}(Z',G) &
\end{tikzcd}$$

□

Lemma 4.9. *With the same setup as in Lemma 4.8, a R-stable G-Higgs bundle (V,θ) (resp. R-stable integrable G-connection (E,∇)) on Z is rigid if and only if the R-stable G-Higgs bundle $\pi^*_{Z/k}(V,\theta)$ (resp. the R-stable integrable G-connection $\pi^*_{Z/k}(E,\nabla)$) on Z' is rigid.*

Proof. This is an analog of [EG20, Lemma 3.4] with respect to the equivalent definition of rigidity in Lemma 4.3. □

Proposition 4.10. *Let S be an arithmetic model for a given complex smooth projective variety X given in Proposition 4.5. There exists a positive integer N, depending on the given data (X,S,G), such that for any closed point $s\in S$ with* $\mathrm{char}\,k(s)>N$, *and any rigid G-Higgs bundle (V_s,θ_s), the inverse Cartier transform $C^{-1}_{\exp}(V'_s,\theta'_s)$ is a rigid integrable G-connection, where $(V'_s,\theta'_s):=\pi^*_{X/k}(V_s,\theta_s)$.*

Proof. The rigidity is defined on stable objects and the equivalence of stability conditions follows from Lemma 4.9 and Theorem 2.11. We only have to check the rigidity.

Let D be an integer bigger than the degree of finite morphisms $\mathcal{M}^{\mathrm{rig}}_*(X_S/S)\to S$, where $*=\mathrm{Dol}$ or dR. Given $s\in S$, let (V_s,θ_s) be a rigid G-Higgs bundle on X_s. Then for any deformation (V_B, θ_B) over an Artinian base B, it corresponds to an element

$$\chi_{\mathrm{Dol}} : \operatorname{Spec} B \to \mathscr{B}'$$

where $\mathscr{B}' := \mathscr{B}(X', G)$, and furthermore, it factors through $\mathscr{B}'^{(D)}$, the nilpotent thickening of 0 of order D.

In the case of SL_n, Esnault–Groechenig showed that there exists an integer N' (depending on D and n) such that when $p > N'$, (twisted) SL_n-Higgs bundles in the preimage of any scheme theoretic point $T \to \mathscr{A}^{(D)}(X', \mathrm{SL}_n)$ under the Hitchin morphism are $[p]$-nilpotent. This property is called OV-admissible in [EG20]. For reductive groups, there is a similar result. We fix a faithful representation $G \hookrightarrow \mathrm{SL}_n$ and obtain a natural commutative diagram

$$\begin{tikzcd}
\mathrm{HIG}_{p-1}(X', G) \arrow[r] \arrow[d, "\mathrm{sd}"] & \mathrm{HIG}_{p-1}(X', \mathrm{SL}_n) \arrow[d, "\mathrm{sd}"] \arrow[dr, "h"] & \\
\mathscr{B}(X', G) \arrow[r] & \mathscr{B}(X', \mathrm{SL}_n) \arrow[r, hook] & \mathscr{A}(X', \mathrm{SL}_n)\,,
\end{tikzcd}$$

where $h : \mathrm{HIG}_{p-1}(X', \mathrm{SL}_n) \to \mathscr{A}(X', \mathrm{SL}_n)$ is the Hitchin morphism. Now we consider the nilpotent thickenings

$$\mathscr{B}'^{(D)} = \mathscr{B}^{(D)}(X', G) \to \mathscr{B}^{(D)}(X', \mathrm{SL}_n) \to \mathscr{A}^{(D)}(X', \mathrm{SL}_n).$$

Thus, any scheme theoretic point $T \to \mathscr{B}'^{(D)}$ will be mapped to scheme theoretic point $T \to \mathscr{A}^{(D)}(X', \mathrm{SL}_n)$. Then any G-Higgs bundle in the preimage of $T \to \mathscr{B}'^{(D)}$ under the spectral data morphism sd will be sent to a SL_n-Higgs bundle included in the preimage of the point $T \to \mathscr{A}^{(D)}(X', \mathrm{SL}_n)$. Now let $N := N'$. As explained in Remark 2.13, when $p > N$, G-Higgs bundles in the preimage of any scheme theoretic point $T \to \mathscr{B}'^{(D)}$ under the spectral data morphism are $[p]$-nilpotent.

Now we choose $m > D$ and assume $p > N$, where N is given as above. If $(E_s, \nabla_s) := C^{-1}_{\exp}(V'_s, \theta'_s)$ is not rigid, then there exists positive dimensional deformation (E_T, ∇_T) over $(X_s)_T$. Assume that either

1. $\chi_{\mathrm{dR}} : T \to \mathscr{B}'$ factors through $\mathscr{B}'^{(m)}$; or
2. $\chi_{\mathrm{dR}} : T \to \mathscr{B}'$ does not factor through $\mathscr{B}'^{(m)}$.

In the first case, there exists a T-family of G-Higgs bundles (V_T, θ_T) such that

$$C^{-1}_{\exp}(V_T, \theta_T) = (E_T, \nabla_T).$$

Since (V'_s, θ'_s) is rigid by Lemma 4.9, (V_T, θ_T) is an infinitesimal deformation of order $\leq D$, which contradicts the condition that χ_{dR} factors through $\mathscr{B}'^{(m)}$.

In the latter case, let $T^{(n)}$ be the n-th order neighborhood of $t \in T$ corresponding to (E_s, ∇_s). We have an induced family

$$(E_{T^{(n)}}, \nabla_{T^{(n)}})$$

over $(X_s)_{T^{(n)}}$, which also induces

$$\chi_{\mathrm{dR}}\colon T^{(n)} \to \mathscr{B}'.$$

There exists $n > m$ such that χ_{dR} factors through $\mathscr{B}'^{(m)}$ but not through $\mathscr{B}'^{(m-1)}$. Moreover,

Since $p > N$, there exists $T^{(n)}$-family of rigid G-Higgs bundles such that

$$\chi_{\mathrm{dR}} = \chi_{\mathrm{Dol}} := \mathrm{sd}((V_{T^{(n)}}, \theta_{T^{(n)}})).$$

Since χ_{Dol} does not factor through $\mathscr{B}'^{(k)}$ for $k < m$, we obtain a strictly order m deformation of (V_s, θ_s), contradicting $m > D$. $\square$

Definition 4.11. Define $n_{\mathrm{dR}}(Z, G)$ (resp. $n_{\mathrm{Dol}}(Z, G)$) to be the number of isomorphism classes of rigid integrable G-connections (resp. G-Higgs bundles) on Z.

Proof of Theorem 1.3. Let S be a nice model given in Proposition 4.5 and take $s \in S$ such that $\operatorname{char} k(s) > N$, where N is the integer given in Proposition 4.10. The properties of nice models imply

$$n_{\mathrm{dR}}(X, G) = n_{\mathrm{dR}}(X_s, G), \quad n_{\mathrm{Dol}}(X, G) = n_{\mathrm{Dol}}(X_s, G).$$

By Lemma 4.9, we know

$$n_{\mathrm{Dol}}(X_s, G) = n_{\mathrm{Dol}}(X'_s, G).$$

Furthermore, Corlette–Simpson correspondence (Theorem 2.9) gives

$$n_{\mathrm{dR}}(X, G) = n_{\mathrm{Dol}}(X, G).$$

Combining the above equalities, we have

$$n_{\mathrm{dR}}(X_s, G) = n_{\mathrm{Dol}}(X'_s, G).$$

The only thing left to show is that rigid integrable G-connections on X_s has nilpotent p-curvature, i.e.,

$$n_{\mathrm{dR}}^{\mathrm{nilp}}(X_s, G) = n_{\mathrm{dR}}(X_s, G),$$

where $n_{\mathrm{dR}}^{\mathrm{nilp}}(X_s, G)$ is the number of stable rigid integrable G-connections on X_s with nilpotent p-curvature. By Lemmas 4.7 and 4.8, we have

$$n_{\mathrm{dR}}^{\mathrm{nilp}}(X_s, G) = n_{\mathrm{Dol}}(X_s, G).$$

Then, the desired equality comes from the following one

$$n_{\mathrm{dR}}^{\mathrm{nilp}}(X_s, G) = n_{\mathrm{Dol}}(X_s, G) = n_{\mathrm{dR}}(X_s, G) \geq n_{\mathrm{dR}}^{\mathrm{nilp}}(X_s, G).$$

This finishes the proof of this theorem. $\square$

References

[AB01] B. Anchouche and I. Biswas. Einstein-Hermitian connections on polystable principal bundles over a compact Kähler manifold. *Amer. J. Math.*, 123(2):207–228, 2001.

[CN20] T. H. Chen and B. C. Ngô. On the Hitchin morphism for higher-dimensional varieties. *Duke Math. J.*, 169(10):1971–2004, 2020.

[CS08] K. Corlette and C. Simpson. On the classification of rank-two representations of quasiprojective fundamental groups. *Compos. Math.*, 144(5):1271–1331, 2008.

[CZ15] T. H. Chen and X. Zhu. Non-abelian Hodge theory for algebraic curves in characteristic p. *Geom. Funct. Anal.*, 25(6):1706–1733, 2015.

[DG70] M. Demazure and P. Gabriel. *Groupes algébriques. Tome I: Géométrie algébrique, généralités, groupes commutatifs.* Masson & Cie, Éditeurs, Paris; North-Holland Publishing Co., Amsterdam, 1970. Avec un appendice *Corps de classes local* par Michiel Hazewinkel.

[DI87] P. Deligne and L. Illusie. Relèvements modulo p^2 et décomposition du complexe de de Rham. *Invent. Math.*, 89(2):247–270, 1987.

[EG18] H. Esnault and M. Groechenig. Cohomologically rigid local systems and integrality. *Selecta Math. (N.S.)*, 24(5):4279–4292, 2018.

[EG20] H. Esnault and M. Groechenig. Rigid connections and F-isocrystals. *Acta Math.*, 225(1):103–158, 2020.

[Esn23a] H. Esnault. *Local systems in algebraic-arithmetic geometry*, volume 2337 of *Lecture Notes in Mathematics*. Springer, Cham, 2023.

[Esn23b] H. Esnault. Some arithmetic properties of complex local systems. *Notices Amer. Math. Soc.*, 70(11):1793–1801, 2023.

[Fær] Joakim Færgeman. Motivic realization of rigid G-local systems on curves and tamely ramified geometric Langlands. arXiv:2405.18268 [math.AG].

[GLSS08] T. L. Gómez, A. Langer, A. H. W. Schmitt, and I. Sols. Moduli spaces for principal bundles in arbitrary characteristic. *Adv. Math.*, 219(4):1177–1245, 2008.

[Kat70] N. Katz. Nilpotent connections and the monodromy theorem: Applications of a result of Turrittin. *Inst. Hautes Études Sci. Publ. Math.*, (39):175–232, 1970.

[Kat72] Nicholas M. Katz. Algebraic solutions of differential equations (p-curvature and the Hodge filtration). *Invent. Math.*, 18:1–118, 1972.

[Kat96] N. Katz. *Rigid local systems*, volume 139 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1996.

[KP22] Christian Klevdal and Stefan Patrikis. G-cohomologically rigid local systems are integral. *Trans. Am. Math. Soc.*, 375(6):4153–4175, 2022.

[KSZ24] G. Kydonakis, H. Sun, and L. Zhao. Logahoric Higgs torsors for a complex reductive group. *Math. Ann.*, 388(3):3183–3228, 2024.

[LS18] A. Langer and C. Simpson. Rank 3 rigid representations of projective fundamental groups. *Compos. Math.*, 154(7):1534–1570, 2018.

[LSZ15] G. Lan, M. Sheng, and K. Zuo. Nonabelian Hodge theory in positive characteristic via exponential twisting. *Math. Res. Lett.*, 22(3):859–879, 2015.

[LSZ19] G. Lan, M. Sheng, and K. Zuo. Semistable Higgs bundles, periodic Higgs bundles and representations of algebraic fundamental groups. *J. Eur. Math. Soc. (JEMS)*, 21(10):3053–3112, 2019.

[McN21] G. J. McNinch. Nilpotent elements and reductive subgroups over a local field. *Algebr. Represent. Theory*, 24(6):1479–1522, 2021.

[OV07] A. Ogus and V. Vologodsky. Nonabelian Hodge theory in characteristic p. *Publ. Math. IHÉS*, 106:1–138, 2007.

[Ram75] A. Ramanathan. Stable principal bundles on a compact Riemann surface. *Math. Ann.*, 213:129–152, 1975.

[Ram96a] A. Ramanathan. Moduli for principal bundles over algebraic curves. I. *Proc. Indian Acad. Sci. Math. Sci.*, 106(3):301–328, 1996.

[Ram96b] A. Ramanathan. Moduli for principal bundles over algebraic curves. II. *Proc. Indian Acad. Sci. Math. Sci.*, 106(4):421–449, 1996.

[Sei00] G. M. Seitz. Unipotent elements, tilting modules, and saturation. *Invent. Math.*, 141(3):467–502, 2000.

[Ser94] J. P. Serre. Sur la semi-simplicité des produits tensoriels de représentations de groupes. *Invent. Math.*, 116(1-3):513–530, 1994.

[Sim90] C. Simpson. Transcendental aspects of the Riemann-Hilbert correspondence. *Illinois J. Math.*, 34(2):368–391, 1990.

[Sim92] C. Simpson. Higgs bundles and local systems. *Publ. Math. IHÉS*, 75:5–95, 1992.

[Sim94a] C. Simpson. Moduli of representations of the fundamental group of a smooth projective variety. I. *Publ. Math. IHÉS*, 79:47–129, 1994.

[Sim94b] C. Simpson. Moduli of representations of the fundamental group of a smooth projective variety. II. *Publ. Math. IHÉS*, 80:5–79, 1994.

[SSW] M. Sheng, H. Sun, and J. Wang. A nonabelian hodge correspondence for principal bundles in positive characteristic. arXiv:2405.09947 [math.AG].

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