Irregular Hodge filtration of hypergeometric differential equations

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Abstract

Fedorov and Sabbah–Yu calculated the (irregular) Hodge numbers of hypergeometric connections. In this paper, we study the irregular Hodge filtrations on hypergeometric connections defined by rational parameters, and provide a new proof of the aforementioned results. Our approach is based on a geometric interpretation of hypergeometric connections, which enables us to show that certain hypergeometric sums are everywhere ordinary on $|\mathbb{G}_{m,\mathbb{F}_p}|$ (i.e., "Frobenius Newton polygon equals to irregular Hodge polygon").

1 Introduction

The primary focus of this article is to investigate the Hodge theoretic properties of confluent hypergeometric differential equations. These differential equations have irregular singularities and are equipped with *irregular Hodge filtrations*, constructed in [32]. The irregular Hodge theory, initiated by Deligne [12], extends the classical Hodge theory and has been developed in a series works such as [31, 24, 42, 15, 33, 32].

Let $n \geq m$ be two non-negative integers, λ a real number, and $\alpha = (\alpha_1, \dots, \alpha_n)$ and $\beta = (\beta_1, \dots, \beta_m)$ two non-decreasing sequences of real numbers in $[0, 1)$. Let S be the scheme $\mathbb{G}_m \backslash \{1\}$ (resp. $\mathbb{G}_m$) if $n = m$ (resp. $n > m$) with coordinate z. The *hypergeometric equation* is the linear differential equation defined by the differential operator

$$\mathrm{Hyp}_\lambda(\alpha;\beta) := \lambda \prod_{i=1}^{n}(z\partial_z - \alpha_i) - z\prod_{j=1}^{m}(z\partial_z - \beta_j). \tag{1.0.0.1}$$

The *hypergeometric connection* $\mathcal{H}yp_\lambda(\alpha;\beta)$ is the associated connection on the complex algebraic variety $S_\mathbb{C}$, see (2.1.1.1). We say that the pair (α, β) is *non-resonant* if $\alpha_i \neq \beta_j$ for any i and j. In this case, the hypergeometric connection $\mathcal{H}yp_\lambda(\alpha;\beta)$ is irreducible and rigid, as seen by combining the works of Beukers–Heckman [9] and Katz [23].

When $n = m$, hypergeometric connections have regular singularities at $0, 1$, and ∞. Simpson demonstrated that rigid irreducible connections on curves with regular singularities, whose eigenvalues of monodromy actions at singularities have norm 1, underlie complex variations of Hodge structure [37, Cor. 8.1]. In this case, Fedorov [16] computed the Hodge numbers associated with the Hodge filtrations of irreducible hypergeometric connections [16], and Martin gave an alternative proof in [25].

When $n > m$, hypergeometric connections are called *confluent*, indicating the merging of singularities, and have a regular singularity at 0 and an irregular singularity at ∞. Sabbah showed in [32, Thm. 0.7] that a rigid irreducible connection on $\mathbb{P}^1$ with real formal exponents at each singular point admits a variation of irregular Hodge structure away from singularities.

For confluent hypergeometric connections, Sabbah and Yu computed the corresponding irregular Hodge numbers [34]. In addition, Castaño Domínguez–Sevenheck [11, Thm. 4.7] and Castaño Domínguez–Reichelt–Sevenheck [10, Thm. 5.8] explicitly calculated the irregular Hodge filtration for $m = 0$ or 1, respectively.

This article focuses on cases where λ, α, and β are rational numbers. We explicitly construct the irregular Hodge filtration $F^{\bullet}_{\mathrm{irr}}$ on hypergeometric connections in Theorem 3.3.1 and provide a uniform method for reproving the results of Fedorov and Sabbah–Yu.

Theorem 1.0.1 (3.3.1)**.** *Suppose (α, β) is non-resonant. We define a map $\theta\colon \{1, \dots, n\} \to \mathbb{R}$ by*

$$\theta(k) = (n-m)\alpha_k + \#\{i \mid \beta_i < \alpha_k\} + (n-k) - \sum_{i=1}^{n} \alpha_i + \sum_{j=1}^{m} \beta_j. \tag{1.0.1.1}$$

Then, up to an $\mathbb{R}$-shift[1]*, the jumps of the irregular Hodge filtration on $\mathcal{H}yp_\lambda(\alpha, \beta)$ occur at $\theta(k)$ and for any $p \in \mathbb{R}$ we have*

$$\operatorname{rk} \operatorname{gr}^p_{F_{\mathrm{irr}}} \mathcal{H}yp_\lambda(\alpha; \beta) = \#\theta^{-1}(p).$$

1.1 Application to Frobenius slopes of hypergeometric sums

Our method has an arithmetic application to the Frobenius slopes of hypergeometric sums, the arithmetic incarnation of hypergeometric connections [23].

Let K be a p-adic field with residue field $\mathbb{F}_p$ containing an element π satisfying $\pi^{p-1} = -p$. Such an element π corresponds to an additive character $\psi : \mathbb{F}_p \to K^\times$ by Dwork's theory [14]. Suppose that (α, β) is non-resonant and that $\alpha_i = \frac{a_i}{p-1}, \beta_j = \frac{b_j}{p-1}$ are in $\frac{1}{p-1}\mathbb{Z}$. Miyatani [27] showed that there exists a unique Frobenius structure φ (up to a scalar) on the analytification of hypergeometric connection $\mathcal{H}yp_{(-1)^{m+np}/\pi^{n-m}}(\alpha; \beta)$ over S_K, which underlies an overconvergent F-isocrystal on S_k (called the *hypergeometric F-isocrystal*). The Frobenius trace of φ at a point $a \in S(\mathbb{F}_q)$ is given by the *hypergeometric sum* $\mathrm{Hyp}(\alpha; \beta)(a)$, defined by

$$\sum_{\substack{x_i, y_j \in \mathbb{F}_q^\times, \\ x_1 \cdots x_n = a y_1 \cdots y_m}} \psi\Big(\mathrm{Tr}\Big(\sum_{i=1}^{n} x_i - \sum_{j=1}^{m} y_j\Big)\Big) \cdot \prod_{i=1}^{n} \omega^{a_i}(\mathrm{Nm}(x_i)) \prod_{j=1}^{m} \omega^{-b_j}(\mathrm{Nm}(y_j)),$$

where $\omega : \mathbb{F}_p^\times \to K^\times$ denotes the Teichmüller lift and $\mathrm{Tr} = \mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_p}$, $\mathrm{Nm} = \mathrm{Nm}_{\mathbb{F}_q/\mathbb{F}_p}$.

Frobenius eigenvalues of φ at a are Weil numbers and have complex absolute valuations $q^{\frac{n+m-1}{2}}$ via an isomorphism $\overline{K} \simeq \mathbb{C}$. When (α, β) is resonant, the above hypergeometric sum can also be written as a sum of n Weil numbers. It is expected that the p-adic valuations of these Frobenius eigenvalues (called *Frobenius slopes*) are related to the (irregular) Hodge filtration. Our construction allows us to show the following result.

Theorem 1.1.1 (4.0.2)**.** *Suppose $n > m$ and that α_i, β_j lie in $\frac{\mathbb{Z}}{p-1} \cap [0, 1)$. For every p-power q and $a \in \mathbb{G}_m(\mathbb{F}_q)$, the multi-set of Frobenius eigenvalues of* $\mathrm{Hyp}(\alpha; \beta)(a)$ *(normalized by* ord_q*) coincides with the multi-set of irregular Hodge numbers $\{\theta(1), \dots, \theta(n)\}$ defined in* (1.0.1.1)*.*

Following [26], we encode the information of the p-adic valuations of Frobenius eigenvalues and (irregular) Hodge numbers into the Newton polygon and the (irregular) Hodge polygon respectively, see Definition 4.0.1.

[1] Our Hodge numbers $\theta(k)$'s are normalized according to the geometric interpretation in Proposition 2.4.1, and is different from those of Fedorov and Sabbah–Yu by a shift.

For crystalline cohomology groups of a smooth proper variety over k, Mazur and Ogus showed that the associated (Frobenius) Newton polygon lies above the Hodge polygon defined by Hodge numbers [26, 8]. For F-isocrystals associated with exponential sums, "Newton above Hodge" type results were studied by Dwork's school. Dwork, Sperber and Wan [14, 38, 39] proved that Kloosterman sums (hypergeometric sums of type $(n, 0)$ with $\alpha = (0, \dots, 0)$) are everywhere ordinary on $|\mathbb{G}_{m,\mathbb{F}_p}|$ (i.e. two polygons coincide for every closed point $a \in |\mathbb{G}_m|$). We use a "Newton above Hodge" result of Adolphson–Sperber [2, 3] and identify their (combinatorial) Hodge polygon for the above hypergeometric sums coincides with the irregular Hodge polygon of hypergeometric connections. Finally, we deduce "Newton equals to Hodge" by a criterion for ordinariness due to Wan [39].

Remark 1.1.2. (i) One may also consider the Frobenius Newton polygon of hypergeometric sums defined by multiplicative characters of orders dividing $p^s - 1$ for a positive integer s. In this case, Adolphson–Sperber showed that the associated Frobenius Newton polygon lies above their (combinatorial) Hodge polygon, which can be viewed as an average of irregular Hodge polygons. However, the associated hypergeometric sums may not be ordinary in the case $s > 1$. There is an example of hypergeometric sums (of type $(n, m) = (2, 0)$), for which the Frobenius Newton polygon lies strictly above Adolphson–Sperber's Hodge polygon for every $a \in |\mathbb{G}_{m,\mathbb{F}_p}|$ [1].

(ii) The ordinariness of hypergeometric sums also fails in the non-confluent case (i.e., $n = m$). For $p = 31$, and the hypergeometric sum defined by $\alpha = (0, 0, 0, 0)$, $\beta = (\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5})$ at $a = 4$ or 17, its Newton polygon (with slope $(\frac{5}{2}, \frac{5}{2}, \frac{9}{2}, \frac{9}{2})$) [13, Appendix A.5][2] strictly lies above the irregular Hodge polygon (with slope $(2, 3, 4, 5)$).

1.2 Strategy of proof

The proof of Theorem 1.0.1 can be reduced to calculating the irregular Hodge filtration on each fiber of $\mathcal{H}yp_\lambda(\alpha, \beta)$. We adopt an approach similar to those used in [18, 35, 30], where the authors calculated the Hodge numbers of motives attached to Kloosterman and Airy moments. The key ingredient of this argument is an (exponentially) geometric interpretation of hypergeometric connections in Proposition 2.3.1.(2). More precisely, there exists a smooth quasi-projective variety X with a regular function $g\colon X \times S \to \mathbb{A}^1$, such that the hypergeometric connections are subquotients of the $\mathscr{D}_S$-module $\mathcal{H}^N \mathrm{pr}_+(\mathcal{O}_{X\times S}, \mathrm{d} + \mathrm{d}g)$, where $N = \dim X$ and pr is the projection $\mathrm{pr}\colon X \times S \to S$. Our construction is motivated by Katz's hypergeometric sums and the function-sheaf dictionary. A related construction can be found in [22].

Through this geometric interpretation, each fiber $\mathcal{H}yp_\lambda(\alpha, \beta)_a$ at $a \in S(\mathbb{C})$ is identified with a subquotient of the twisted de Rham cohomology of the pair $(X, g_a := g\,|_{\mathrm{pr}_z^{-1}(a)})$, i.e., the hypercohomology of the twisted de Rham complex $(\Omega_X^\bullet, \mathrm{d} + \mathrm{d}g_a)$. Then, we reduce to calculate the irregular Hodge filtration on the twisted de Rham cohomology of the pair (X, g_a) (up to a shift).

The irregular Hodge filtration on the twisted de Rham cohomology of the pairs (X, g_a) has been studied by Yu [42]. In the context of our case, we can select $X = \mathbb{G}_m^{n+m-1}$ and g_a as a Laurent polynomial with good properties, see Proposition 2.3.1. Under these assumptions, Yu showed that the irregular Hodge filtration on $\mathrm{H}_{\mathrm{dR}}^{n+m-1}(X, g_a)$ can be calculated by the Newton polyhedron filtration on the Newton polytope $\Delta(g_a)$ (3.1.0.4). This identification enables us to calculate via a combinatorial approach, leading to a fiber-wise version of Theorem 1.0.1 as follows:

Theorem 1.2.1 (3.3.3)**.** *Up to an $\mathbb{R}$-shift, the jumps of the irregular Hodge filtration $F_{\mathrm{irr}}^\bullet$ on the fiber $\mathcal{H}yp(\alpha;\beta)_a$ occur at $\theta(k)$ from (1.0.1.1) for $1 \leq k \leq n$. Moreover, we have $\dim \mathrm{gr}^p_{F_{\mathrm{irr}}} \mathcal{H}yp(\alpha;\beta)_a = \#\theta^{-1}(p)$ for any $p \in \mathbb{R}$.*

[2] In *loc. cit*, the Frobenius slopes are normalized and are different from our convention by a shift of 2.

Moreover, our calculation allows us to answer a question of Katz [23, 6.3.8] on the comparison between modified hypergeometric $\mathscr{D}$-modules and hypergeometric connections in the resonant case (see Proposition 2.4.6) when the parameters are rational.

1.3 Organization of this article

The article is organized as follows. We present a geometric interpretation of hypergeometric connections in Section 2. Section 3 is devoted to the proof of Theorem 1.2.1 and Theorem 1.0.1. In Section 4, we study hypergeometric sums defined by multiplicative characters of orders dividing $p-1$ and prove that they are ordinary (Theorem 1.1.1).

2 Hypergeometric connections

In this section, we give an (exponentially) geometrical interpretation of the hypergeometric connections in Propositions 2.3.1 and 2.4.1. We work with varieties over $\mathbb{C}$ in Sections 2 and 3.

2.1 Review of hypergeometric connections following [23]

2.1.1. Hypergeometric connections. Let $n \geq m$ be two integers ≥ 0, $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\beta = (\beta_1, \ldots, \beta_j)$ two sequences of non-decreasing rational numbers (and we don't require that they lie in $[0, 1)$ as in § 1), and $\lambda \in \mathbb{Q}$. Let $\mathscr{D}_S$ be the sheaf of differential operator on S (§ 1). Then, the hypergeometric connection $\mathcal{H}yp_\lambda(\alpha;\beta)$ on S is defined by (1.0.0.1)

$$\mathscr{D}_S/\mathrm{Hyp}_\lambda(\alpha;\beta). \tag{2.1.1.1}$$

By [23, (3.1)], one has for $\gamma \in \mathbb{Q}$ that

$$\mathcal{H}yp_\lambda(\alpha;\beta) \otimes (\mathcal{O}, \mathrm{d} + \gamma \tfrac{\mathrm{d}z}{z}) \simeq \mathcal{H}yp_\lambda(\alpha+\gamma;\beta+\gamma), \tag{2.1.1.2}$$

where $\alpha+\gamma$ (resp. $\beta+\gamma$) is the sequence consisting of $\alpha_i+\gamma$ (resp. $\beta_j+\gamma$). Furthermore, one has for $\mu \in \mathbb{Q}^\times$ that

$$[x \mapsto \mu \cdot x]^+ \mathcal{H}yp_\lambda(\alpha;\beta) \simeq \mathcal{H}yp_{\lambda/\mu}(\alpha;\beta). \tag{2.1.1.3}$$

Thanks to the above relations, we can often assume that $\lambda = 1$ and $\alpha_1 = 0$. For simplicity, we denote by $\mathcal{H}yp(\alpha;\beta)$ the connection $\mathcal{H}yp_1(\alpha;\beta)$.

When the pair (α,β) is non-resonant, i.e., $\alpha_i - \beta_j \notin \mathbb{Z}$ for any i, j, Katz showed in [23, Prop. 3.2] that $\mathcal{H}yp(\alpha;\beta)$ is irreducible, and only depends on $\alpha \bmod \mathbb{Z}$ and $\beta \bmod \mathbb{Z}$. In this case, we may assume that α and β are two non-decreasing sequences of rational numbers in $[0, 1)$.

2.1.2. Modified hypergeometric $\mathscr{D}$-modules. Given a morphism g between smooth varieties, for bounded complex of holonomic algebraic $\mathscr{D}$-modules, following [18, App. A.1], we denote by g^+, g_+, and $g_\dagger$ the derived pullback functor, the pushforward functor, and the pushforward with compact support functor respectively. The k-th cohomology of a complex K is denoted by $\mathcal{H}^k(K)$.

Let mult: $\mathbb{G}_m \times \mathbb{G}_m \to \mathbb{G}_m$ be the product map. The convolution functors $\star_*$ and $\star_!$ on $\mathbb{G}_m$ are defined, for two objects M and N of $\mathrm{D}^b(\mathscr{D}_{\mathbb{G}_m})$ by

$$M \star_* N := \mathrm{mult}_+(M \boxtimes N) \text{ and } M \star_! N := \mathrm{mult}_\dagger M \boxtimes N$$

respectively. These convolution functors are associative and commutative. Moreover, the duality functor $\mathbb{D}$ interchanges $\star_!$ and $\star_*$.

Definition 2.1.3. Let α and β be two sequences of rational numbers. For $? \in \{!, *\}$, the convolution

$$\mathcal{H}yp(\alpha_1; \varnothing) \star_? \cdots \star_? \mathcal{H}yp(\alpha_n; \varnothing) \star_? \mathcal{H}yp(\varnothing; \beta_1) \star_? \cdots \star_? \mathcal{H}yp(\varnothing; \beta_m)$$

is a holonomic $\mathscr{D}_{\mathbb{G}_m}$-module [23, (6.3.6)]. We denote it by $\mathcal{H}yp(?; \alpha; \beta)$ and call it *modified hypergeometric* $\mathscr{D}$*-module.*

The above two modified hypergeometric $\mathscr{D}$-modules are generally not isomorphic to the hypergeometric connections in general. When (α, β) is non-resonant, the natural map

$$\mathcal{H}yp(!; \alpha; \beta) \to \mathcal{H}yp(*; \alpha; \beta) \tag{2.1.3.1}$$

is an isomorphism, as seen by using an argument similar to those in [23, Thm. 8.4.2(5)] and [27, Prop. 3.3.3]. In this case, both modified hypergeometric $\mathscr{D}_{\mathbb{G}_m}$-modules are isomorphic to the hypergeometric connection $\mathcal{H}yp(\alpha; \beta)$ by [23, (5.3.1)].

2.2 The Newton polytope of a Laurent polynomial

We study the Newton polytope of a Laurent polynomial appearing in the geometric interpretation of hypergeometric connections in Proposition 2.4.1.

Definition 2.2.1. Let N be a positive integer and $g(z_1, \cdots, z_N) = \sum_{\tau \in \mathbb{Z}^N} c(\tau) z^\tau$ be a Laurent polynomial in variables $z_1, \ldots, z_N$, with $z^\tau = \prod_{i=1}^N z_i^{\tau_i}$ for $\tau = (\tau_1, \cdots, \tau_N)$.

(1). The support of g is the subset $\mathrm{Supp}(g) = \{\tau \mid c(\tau) \neq 0\}$ of $\mathbb{Z}^N$.

(2). The *Newton polytope* $\Delta(g)$ is the convex hull of the set $\mathrm{Supp}(g) \cup \{0\}$ in $\mathbb{R}^N$.

(3). The Laurent polynomial g is called *non-degenerate* with respect to $\Delta(g)$ (or simply non-degenerate) if for each face $\sigma \subset \Delta(g)$ not passing through 0, the Laurent polynomial $g_\sigma := \sum_{\tau \in \sigma \cap \mathbb{Z}^N} c(\tau) z^\tau$ has no critical point in $(\mathbb{C}^\times)^N$.

Let $n \geq m \geq 0$ and $d \geq 1$ be three integers, and $f \colon \mathbb{G}_m^{n+m} \to \mathbb{A}^1$ the Laurent polynomial

$$f : (x_2, \ldots, x_n, y_1, \ldots, y_m, z) \mapsto \sum_{i=2}^n x_i^d - \sum_{j=1}^m y_j^d + z \cdot \frac{\prod_{j=1}^m y_j^d}{\prod_{i=2}^n x_i^d}, \tag{2.2.1.1}$$

and $\mathrm{pr}_z \colon \mathbb{G}_m^{n+m} \to \mathbb{G}_m$ the projection onto the z-coordinate. For $a \in \mathbb{C}^\times$, we set $f_a = f\,|_{\mathrm{pr}_z^{-1}(a)}$.

We denote by $\{u_i, v_j\}_{2 \leq i \leq n, 1 \leq j \leq m}$ the coordinates in $\mathbb{R}^{n+m-1}$, and identify a monomial $\prod_i x_i^{a_i} \cdot \prod_j y_j^{b_j}$ with a lattice point $(a_i, b_j) \in \mathbb{Z}^{n+m-1} \subset \mathbb{R}^{n+m-1}$.

Lemma 2.2.2. *Assume that $n > m = 0$ and $a \in \mathbb{C}^\times$.*

(1). The Laurent polynomial f_a is convenient, i.e., the origin is in the interior of $\Delta(f_a)$.

(2). The Newton polytope $\Delta(f_a)$ is defined by

$$h_{n+1} := \sum_{i=2}^n u_i \leq d \quad \text{and} \quad h_{i_0} := \sum_{i=2}^n u_i - (n-m) u_{i_0} \leq d, \quad 2 \leq i_0 \leq n. \tag{2.2.2.1}$$

(3). The Laurent polynomial f_a is non-degenerate with respect to $\Delta(f_a)$.

Proof. (1) Let P_i for $2 \leq i \leq n$, and R be the points in $\mathbb{Z}^{n-1}$ corresponding to x_i^d and $1/\prod x_i^d$ respectively. Observe that 0 is an interior point of the Newton polytope because $0 = \frac{1}{n}(\sum P_i + R)$.

(2) A face $\sigma \subset \Delta(f_a)$ of dimension $n-2$ must pass through $n-1$ points among $\{P_i, R\}$. So either $R \notin \sigma$ or there exists a $P_{i_0} \notin \sigma$. In the first case, the face lies on the hyperplane defined by the equation $h_{n+1} = d$. In the latter case, the face lies on the hyperplane defined by the equations $h_{i_0} = d$.

(3) Let σ be a face which does not pass through 0. Since the support of f_a has n points, it must pass through at most $n-1$ points in $\mathrm{Supp}(f_a)$. Let $I \subset \{2, \ldots, n\}$ be a subset of the indices. Then $f_{a,\sigma}$ is either

$$f_{a,\sigma} = \sum_{i \in I} x_i^d, \quad \text{or} \quad f_{a,\sigma} = \sum_{i \in I} x_i^d + \frac{a}{\prod_{i=2}^n x_i^d}, \quad |I| \leq n-2.$$

We can check that they are smooth on $\mathbb{G}_m^{n-1}$. So f_a is non-degenerate. □

Lemma 2.2.3. *Assume that $n > m \neq 0$ and $a \in \mathbb{C}^\times$.*

(1). The cone $\mathbb{R}_{\geq 0} \cdot \Delta(f_a)$ is defined by

$$u_i + v_j \geq 0, \quad v_j \geq 0$$

for $i = 2, \ldots, n$ and $j = 1, \ldots, m$,

(2). The Newton polytope $\Delta(f_a)$ is defined by

$$u_i + v_j \geq 0, \quad v_j \geq 0, \quad h_{n+1} := \sum u_i + \sum v_j \leq d$$

and

$$h_{i_0} := \sum_i u_i + \sum_j v_j - (n-m)u_{i_0} \leq d, \quad 2 \leq i_0 \leq n. \tag{2.2.3.1}$$

(3). The Laurent polynomial f_a is non-degenerate with respect to $\Delta(f_a)$ [3].

Proof. Let P_i and Q_j be the points in $\mathbb{Z}^{n+m-1}$ corresponding to monomials x_i^d and y_j^d for $2 \leq i \leq n$ and $1 \leq j \leq m$ respectively, and R the lattice point corresponding to $\prod_{j=1}^m y_j^d / \prod_{i=2}^n x_i^d$. In this case, the origin 0 is not an interior point of the Newton polytope. So $\Delta(f_a)$ has $(n+m+1)$-many vertices. To determine a face of dimension $n+m-2$, we need to choose $(n+m-1)$-many points among $\{P_i, Q_j, R\}$.

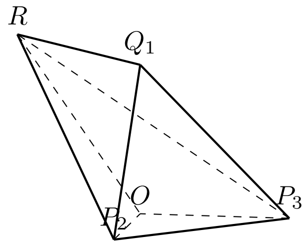
(1) For the first part, it suffices to determine faces $\sigma \subset \Delta(f_a)$ with dimensions $n+m-2$ containing 0.

- If σ does not pass through R, it contain $(n+m-2)$ distinct points in $\{P_i, Q_j\}$. In this case, σ misses one point Q_{j_0}, and lies on the hyperplane $v_{j_0} = 0$. Otherwise, σ misses one point P_{i_0}. Hence, the hyperplane is given by the equation $u_{i_0} = 0$. Therefore, R and P_{i_0} lie on the two sides of the hyperplane respectively, which is absurd.
- If σ passes through R, it contains $(n+m-3)$ distinct points in $\{P_i, Q_j\}$. In this case, σ has to miss one P_{i_0} and one Q_{j_0}, and lies on the hyperplane $u_{i_0} + v_{j_0} = 0$. Otherwise, σ misses two $P_{i_0}, P_{i_0'}$ or $Q_{j_0}, Q_{j_0'}$. So σ lies on the hyperplane $u_{i_0} - u_{i_0'} = 0$ or $v_{j_0} - v_{j_0'} = 0$. However, the points $P_{i_0}, P_{i_0'}$ or $Q_{j_0}, Q_{j_0'}$ lie be on different sides of the hyperplane $u_{i_0} - u_{i_0'} = 0$ or $v_{j_0} - v_{j_0'} = 0$, which contradicts the definition of σ.

[3] In [5, Lem. 3.6], there is an alternative way of proving that f_a is non-degenerate in this setting.

(2) For the second part, it suffices to determine faces of dimension $n+m-2$ that do not pass through the origin.

- If $R \notin \sigma$, then σ contains all points P_i and Q_j. In this case, σ lies on the hyperplane $\sum u_i + \sum v_j = d$.
- If $R \in \sigma$, then σ contains $n+m-2$ points among $\{P_i, Q_j\}$. In this case, σ misses one P_{i_0}, and lies on the hyperplane $h_{i_0} = d$. Otherwise, it misses one Q_{j_0} and lies on the hyperplane $\sum_{i=2}^n u_i + \sum_{j=1}^m v_j + (n-m)v_{j_0} = d$. However, the points 0 and Q_{j_0} are on different sides of the hyperplane.



(3) Let σ be a face which does not pass through 0. Since the support of f_a has $n+m$ points, it must pass through at most $n+m-1$ points in $\mathrm{Supp}(f_a)$. Let $I \subset \{2, \ldots, n\}$ and $J \subset \{1, \ldots, m\}$ be two subsets of the indices. Then $f_{a,\sigma}$ is either

$$f_{a,\sigma} = \sum_{i\in I} x_i^d - \sum_{j \in J} y_j^d, \quad \text{or} \quad f_{a,\sigma} = \sum_{i\in I} x_i^d - \sum_{j\in J} y_j^d + \frac{a\prod y_j^d}{\prod x_i^d}, \quad \text{for } |I|+|J| \leq n+m-2.$$

We can check that they are smooth on $\mathbb{G}_m^{n+m-1}$. So f_a is non-degenerate. □

Lemma 2.2.4. *Assume that $n = m$ and $a \in \mathbb{C}^\times$.*

(1). The cone $\mathbb{R}_{\geq 0} \cdot \Delta(f_a)$ is defined by

$$u_i + v_j \geq 0, \quad v_j \geq 0$$

for $i = 2, \ldots, n$ and $j = 1, \ldots, m$,

(2). The Newton polytope $\Delta(f_a)$ is defined by

$$u_i + v_j \geq 0, \quad v_j \geq 0, \quad \text{and} \quad h_{n+1} := \sum u_i + \sum v_j \leq d. \tag{2.2.4.1}$$

(3). The Laurent polynomial f_a is non-degenerate with respect to $\Delta(f_a)$ if $a \neq 1$.

Proof. We use the same notation as in Lemma 2.2.3. The proof of the first assertion is the same as that in Lemma 2.2.3. The second assertion follows from the observation that the points $\{P_i, Q_j, R\}$ all lie on the hyperplane $\sum u_i + \sum v_j - d = 0$.

Let σ be the face passing through $\{P_i, Q_j, R\}$. If a face τ of $\Delta(f_a)$ does not contain 0, it is a face of σ. One can check that if τ is a proper face of σ, there is no solution for the system of equations

$$f_{a,\tau} = \partial_{x_i} f_{a,\tau} = \partial_{y_j} f_{a,\tau} = 0.$$

If $\tau = \sigma$, the system of equations

$$f_a = \partial_{x_i} f_a = \partial_{y_j} f_a = 0$$

has solutions in $\mathbb{G}_m^{n+m-1}$ if and only if $a = 1$. So f_a is non-degenerate with respect to $\Delta(f_a)$ if $a \neq 1$. □

Remark 2.2.5. The volume of $\Delta(f_a)$ is $\frac{d^{n+m-1}n}{(n+m-1)!}$. In fact, the Newton polytope can be decomposed into n-copies $n+m-1$-simplexes, and each of them has volume $\frac{d^{n+m-1}}{(n+m-1)!}$.

2.3 Geometric interpretations

We present geometric interpretations of hypergeometric connections here. Let d be a common denominator of α_i and β_j, and set $a_i = d\cdot\alpha_i$ and $b_j = d\cdot\beta_j$. To α_i (resp. β_j), we associate the character $\chi_i\colon \mu_d \to \mathbb{C}^\times$ (resp. ρ_j) which sends ζ_d to $\zeta_d^{a_i}$ (resp. $\zeta_d^{b_j}$). Set

$$\chi\times\rho = \chi_1\times\ldots\times\chi_n\times\rho_1^{-1}\times\ldots\times\rho_m^{-1}, \quad \widetilde{\chi}\times\rho = \chi_2\times\ldots\times\chi_n\times\rho_1^{-1}\times\ldots\times\rho_m^{-1} \tag{2.3.0.1}$$

as products of these characters.

Now we introduce two diagrams as follows:

- Let $\mathbb{G}_m^{n+m}$ be the torus with coordinates x_i, y_j for $1\le i\le n$ and $1\le j\le m$. The group μ_d^{n+m} acts on $\mathbb{G}_m^{n+m}$ by multiplication. We consider the diagram

$$\begin{array}{ccc} & \mathbb{G}_m^{n+m} & \\ {}^{\sigma}\swarrow & & \searrow^{\varpi} \\ \mathbb{A}^1 & & \mathbb{G}_m \end{array} \tag{2.3.0.2}$$

  where $\sigma(x_i,y_j) = \sum_{i=1}^n x_i^d - \sum_{j=1}^m y_j^d$, and $\varpi(x_i,y_j) = \prod_{i=1}^n x_i^d / \prod_{j=1}^m y_j^d$.

- Let $\mathbb{G}_m^{n+m}$ be the torus with coordinates z, x_i, y_j for $2\le i\le n$ and $1\le j\le m$, and S be $\mathbb{G}_m$ (resp. $\mathbb{G}_m\backslash\{1\}$) if $n\neq m$ (resp. $n=m$). The group $G = \mu_d^{n+m-1}$ acts on coordinates x_i's and y_j's by multiplication. We consider the diagram

$$\begin{array}{ccccc} & \mathbb{G}_m^{n+m} & \hookleftarrow & U := S\times\mathbb{G}_m^{n+m-1} & \\ {}^{f}\swarrow & & \searrow^{\mathrm{pr}_z} & & \searrow^{\mathrm{pr}_z} \\ \mathbb{A}^1 & & & \mathbb{G}_m & \hookleftarrow\; S \end{array} \tag{2.3.0.3}$$

  where pr_z is the projection on the z-coordinate and f is the Laurent polynomial

$$\sum_{i=2}^n x_i^d - \sum_{j=1}^m y_j^d + z\cdot\frac{\prod_{j=1}^m y_j^d}{\prod_{i=2}^n x_i^d},$$

  defined in (2.2.1.1).

Let $\mathcal{E}^z = (\mathcal{O}, \mathrm{d}+\mathrm{d}z)$ be the exponential $\mathscr{D}$-module on $\mathbb{A}^1_z$. For a regular function $f\colon X\to\mathbb{A}^1_z$, we denote by $\mathcal{E}^f$ the connection $(\mathcal{O}_X, \mathrm{d}+\mathrm{d}f)$ on X.

Proposition 2.3.1. *Let α and β be as above.*

(1). The complex $\varpi_?\mathcal{E}^\sigma$ is concentrated in degree 0 for $?\in\{\dagger,+\}$, and we have isomorphisms of $\mathscr{D}$-modules

$$\mathcal{H}yp(*;\alpha;\beta)\simeq(\varpi_+\mathcal{E}^\sigma)^{(\mu_d^{n+m},\chi\times\rho)} \quad \text{and} \quad \mathcal{H}yp(!;\alpha;\beta)\simeq(\varpi_\dagger\mathcal{E}^\sigma)^{(\mu_d^{n+m},\chi\times\rho)},$$

where the exponent $(\mu_d^{n+m},\chi\times\rho)$ means taking the $\chi\times\rho$-isotypic component with respect to the action of μ_d^{n+m}.

(2). If $\alpha_1 = 0$, we have

$$\mathcal{H}yp(*;\alpha;\beta) \simeq (\mathcal{H}^0\mathrm{pr}_{z+}\mathcal{E}^f)^{(G,\widetilde{\chi}\times\rho)} \quad \textit{and} \quad \mathcal{H}yp(!;\alpha;\beta) \simeq (\mathcal{H}^0\mathrm{pr}_{z\dagger}\mathcal{E}^f)^{(G,\widetilde{\chi}\times\rho)}.$$

Proof. The case of $\mathcal{H}yp(!;\alpha;\beta)$ can be deduced from the case of $\mathcal{H}yp(*;\alpha;\beta)$ by applying the duality functor. So, we only prove the latter case.

(1) Assume that $(n,m) = (1,0)$. Then $\sigma\colon \mathbb{G}_{m,x_1} \to \mathbb{A}^1$ is the map $x_1 \mapsto x_1^d$ and $\varpi\colon \mathbb{G}_{m,x_1} \to \mathbb{G}_{m,z}$ is the d-th power map. So by the identity $\varpi_+\mathcal{O}_{\mathbb{G}_m} = \bigoplus_{i=0}^{d-1}\left(\mathcal{O}_{\mathbb{G}_m}, \mathrm{d} + \frac{i}{d}\frac{\mathrm{d}z}{z}\right)$ and the projection formula, we have

$$(\varpi_+\mathcal{E}^\sigma) = \mathcal{E}^z \otimes (\varpi_+\mathcal{O}_{\mathbb{G}_m}) = \bigoplus_{i=0}^{d-1} \mathcal{E}^z \otimes \left(\mathcal{O}_{\mathbb{G}_m}, \mathrm{d} + \tfrac{i}{d}\tfrac{\mathrm{d}z}{z}\right),$$

which is concentrated in degree 0. Taking the isotypic component, we have

$$\begin{aligned}(\varpi_+\mathcal{E}^\sigma)^{(\mu_d^{n+m},\chi\times\rho)} &= (\varpi_+\mathcal{E}^{x_1^d})^{(\mu_d,\chi_1)} = \mathcal{E}^z \otimes (\varpi_+\mathcal{O}_{\mathbb{G}_m})^{(\mu_d,\chi_1)} \\ &= (\mathcal{O}_{\mathbb{G}_m}, \mathrm{d} + \mathrm{d}z + \alpha_1\tfrac{\mathrm{d}z}{z}) = \mathcal{H}yp(*;\alpha_1;\varnothing)\end{aligned}$$

in the case where $(n,m) = (1,0)$. The proof of the case where $(n,m) = (0,1)$ is similar. In general, we use the induction on $n+m$. The proof follows from the following lemma.

Lemma 2.3.2. *Let α, α, β and β' be four sequences of rational numbers with common denominator d, whose lengths are n, n', m and m' respectively. We denote by $\chi_i, \chi_i', \rho_j, \rho_j'$ characters of μ_d corresponding to $\alpha_i, \alpha_i', \beta_j, \beta_j'$ respectively. Let σ, and ϖ (resp. σ' and φ') be the maps for (n,m) (resp. (n',m')) in the diagram (2.3.0.2).*

Suppose that $(\varpi_+\mathcal{E}^\sigma)$ and $(\varpi'_+\mathcal{E}^{\sigma'})$ are concentrated in degree 0, and there are isomorphisms of $\mathscr{D}$-modules

$$\mathcal{H}yp(*;\alpha;\beta) \simeq (\varpi_+\mathcal{E}^\sigma)^{(\mu_d^{n+m},\chi\times\rho)} \quad \textit{and} \quad \mathcal{H}yp(*;\alpha';\beta') \simeq (\varpi_+\mathcal{E}^{\sigma'})^{(\mu_d^{n+m},\chi'\times\rho')}.$$

Then $((\varpi\cdot\varpi')_+\mathcal{E}^{\sigma\boxplus\sigma'})$ is also concentrated in degree 0, and we have an isomorphism of $\mathscr{D}$-modules

$$\mathcal{H}yp(*;\alpha,\alpha';\beta,\beta') \simeq ((\varpi\cdot\varpi')_+\mathcal{E}^{\sigma\boxplus\sigma})^{(\mu_d^{n+n'+m+m'},\chi\times\chi'\times\rho\times\rho')}$$

where $\varpi\cdot\varpi' = \mathrm{mult}\circ(\varpi\times\varpi')$, pr and pr' are the projections from $\mathbb{G}_m^{n+n'+m+m'}$ to $\mathbb{G}_m^{n+m}$ and $\mathbb{G}_m^{n'+m'}$ respectively, and $\sigma\boxplus\sigma' = \sigma\circ\mathrm{pr} + \sigma'\circ\mathrm{pr}'$ is the Thom-Sebastiani sum.

Proof of Lemma 2.3.2. This proof of this lemma is essentially that of [23, Lem. 5.4.3]. Notice that the exterior product $\mathcal{E}^\sigma \boxtimes \mathcal{E}^{\sigma'}$ is $\mathcal{E}^{\sigma\boxplus\sigma'}$. Then

$$\begin{aligned}(\varpi_+\mathcal{E}^\sigma) \star_* (\varpi'_+\mathcal{E}^{\sigma'}) &= \mathrm{mult}_+((\varpi_+\mathcal{E}^\sigma)\boxtimes(\varpi'_+\mathcal{E}^{\sigma'})) \\ &= \mathrm{mult}_+(\varpi\times\varpi')_+(\mathcal{E}^\sigma\boxtimes\mathcal{E}^{\sigma'}) = (\varpi\cdot\varpi')_+\mathcal{E}^{\sigma\boxplus\sigma'}.\end{aligned}$$

By Künneth formula [20, Prop. 1.5.28(i) and Prop. 1.5.30], we conclude that $(\varpi\cdot\varpi')_+\mathcal{E}^{\sigma\boxplus\sigma'}$ is again concentrated in degree 0. We finish the proof by taking the corresponding isotypic components. ☐

(2) Since $\alpha_1 = 0$, the character χ_1 is trivial. So we have

$$\begin{aligned}(\varpi_+\mathcal{E}^\sigma)^{(\mu_d^{n+m},\chi\times\rho)} &= \left(\left(x_1\cdot\prod_{i=2}^n x_i^d/\prod_{j=1}^m y_j^d\right)_+ \mathcal{E}^{x_1+\sum_{i=2}^m x_i^d - \sum_j y_j^d}\right)^{(1\times G, 1\times\widetilde{\chi}\times\rho)} \\ &= (\mathrm{pr}_{z+}\mathcal{E}^f)^{(G,\widetilde{\chi}\times\rho)},\end{aligned} \tag{2.3.2.1}$$

where we performed a change of variable $z = x_1 \cdot \prod_{i=2}^n x_i^d / \prod_{j=1}^m y_j^d$ to get rid of the variable x_1 in the last isomorphism. Because $(\varpi_+\mathcal{E}^\sigma)$ is concentrated in degree 0, so is $(\mathrm{pr}_{z+}\mathcal{E}^f)^{(G,\widetilde{\chi}\times\rho)}$. □

Corollary 2.3.3. *Assume that (α,β) is non-resonant and $\alpha_1 = 0$. Then, the natural map*

$$(\mathcal{H}^0\mathrm{pr}_{z\dagger}\mathcal{E}^f)^{(G,\widetilde{\chi}\times\rho)} \to (\mathcal{H}^0\mathrm{pr}_{z+}\mathcal{E}^f)^{(G,\widetilde{\chi}\times\rho)}$$

is an isomorphism of $\mathscr{D}_{\mathbb{G}_m}$-modules. In particular, for $a \in S(\mathbb{C})$, the forget-support map

$$\mathrm{H}^{n+m-1}_{\mathrm{dR},c}(\mathbb{G}_m^{n+m-1}, f_a) \to \mathrm{H}^{n+m-1}_{\mathrm{dR}}(\mathbb{G}_m^{n+m-1}, f_a)$$

is an isomorphism.

Proof. Using induction on the size of α and β, one can verify that the diagram

$$\begin{array}{ccc}
\mathcal{H}yp(!;\alpha;\beta) & \xrightarrow{\simeq} & \mathcal{H}yp(*;\alpha;\beta) \\
\downarrow\simeq & & \downarrow\simeq \\
(\mathcal{H}^0\varpi_\dagger\mathcal{E}^\sigma)^{(\mu_d^{n+m},\chi\times\rho)} & \longrightarrow & (\mathcal{H}^0\varpi_+\mathcal{E}^\sigma)^{(\mu_d^{n+m},\chi\times\rho)}|_S \\
\downarrow\simeq & & \downarrow\simeq \\
(\mathcal{H}^0\mathrm{pr}_{z\dagger}\mathcal{E}^f)^{(G,\widetilde{\chi}\times\rho)} & \longrightarrow & (\mathcal{H}^0\mathrm{pr}_{z+}\mathcal{E}^f)^{(G,\widetilde{\chi}\times\rho)}
\end{array}$$

is commutative, where the horizontal morphisms are the natural morphisms, the two upper vertical morphisms are those from Proposition 2.3.1.(1), and the two lower vertical morphisms are (2.3.2.1). So, we deduce the isomorphism

$$(\mathcal{H}^0\mathrm{pr}_{z\dagger}\mathcal{E}^f)^{(G,\widetilde{\chi}\times\rho)} \to (\mathcal{H}^0\mathrm{pr}_{z+}\mathcal{E}^f)^{(G,\widetilde{\chi}\times\rho)}.$$

At last, we take the non-characteristic inverse image along $a\colon \mathrm{Spec}(\mathbb{C}) \to \mathbb{G}_m$, and the base change theorem [20, Thm. 1.7.3 & Prop. 1.5.28] to conclude the isomorphism of twisted de Rham cohomologies. □

Remark 2.3.4. When (α,β) is non-resonant and $\alpha_1 = 0$, we deduce from Proposition 2.3.1 the isomorphism

$$[z \mapsto (-1)^{n-m}z]^+\mathcal{H}yp(\alpha,\beta) \simeq (\mathcal{H}^0\mathrm{pr}_{z+}\mathcal{E}^{-f})^{(G,\widetilde{\chi}\times\rho)},$$

by performing a change of variable by sending x_i and y_j to $-x_i$ and $-y_j$ respectively in the diagram (2.3.0.3). According to (2.1.1.3), the first term in the above is $\mathcal{H}yp_{(-1)^{n-m}}(\alpha;\beta)$. In particular, the results in Corollary 2.3.3 remain valid if we replace f with $-f$.

2.4 Explicit cyclic vectors for hypergeometric connections

We present explicit cyclic vectors for $\mathcal{H}yp(\alpha;\beta)$ in terms of sections of some subquotients of some relative de Rham cohomology equipped with their Gauss–Manin connections. This point of view will be used in the computation of Hodge filtrations in Section 3.

Recall that d is an integer such that $a_i = d\alpha_i$ and $b_j = d\beta_j$ are integers for all i, j, and we take notation from (2.3.0.3). When (α,β) is non-resonant and $\alpha_1 = 0$, there exists an isomorphism between the hypergeometric connection $\mathcal{H}yp(\alpha;\beta)$ and the relative de Rham cohomology $\mathcal{H}^{n+m-1}_{\mathrm{dR}}(U/S, f)^{(G,\widetilde{\chi}\times\rho)}$ equipped with the Gauss–Manin connection by Proposition 2.3.1.

Proposition 2.4.1. *Suppose that $\alpha_1 = 0$ and (α, β) is non-resonant. The relative de Rham cohomology $\mathcal{H}^{n+m-1}_{\mathrm{dR}}(U/S, f)^{(G,\widetilde{\chi}\times\rho)}$ admits a cyclic vector, defined by the cohomology class of the differential form*

$$\omega = \prod_{i=2}^{n} x_i^{a_i} \cdot \prod_{j=1}^{m} y_j^{-b_j} \frac{\mathrm{d}x_2}{x_2} \cdots \frac{\mathrm{d}x_n}{x_n} \frac{\mathrm{d}y_1}{y_1} \cdots \frac{\mathrm{d}y_m}{y_m}.$$

Remark 2.4.2. Under the above assumption, the isomorphism class of $\mathcal{H}yp(\alpha;\beta)$ depends only on the congruent classes of α, β modulo $\mathbb{Z}$. Then, any differential form

$$\omega = \prod_{i=2}^{n} x_i^{u_i} \cdot \prod_{j=1}^{m} y_j^{-v_j} \frac{\mathrm{d}x_2}{x_2} \cdots \frac{\mathrm{d}x_n}{x_n} \frac{\mathrm{d}y_1}{y_1} \cdots \frac{\mathrm{d}y_m}{y_m},$$

satisfying $u_i \equiv a_i, v_j \equiv b_j$ modulo d, is a cyclic vector of $\mathcal{H}^{n+m-1}_{\mathrm{dR}}(U/S, f)^{(G,\widetilde{\chi}\times\rho)}$.

Proof. The morphism $\mathrm{pr}_z \colon U \to S$ is smooth (2.3.0.3). It follows that the relative de Rham cohomologies $\mathcal{H}^i_{\mathrm{dR}}(U/S, f)$ are equipped with the Gauss-Manin connections $D := \nabla_{z\partial_z}$, given by

$$\nabla_{z\partial_z}\omega = z\partial_z\omega + z\partial_z(f)\omega \tag{2.4.2.1}$$

for $0 \le i \le n+m-1$. By Lemmas 2.2.2, 2.2.3, and 2.2.4, the Laurent polynomial $f_a := f|_{\mathrm{pr}_z^{-1}(a)}$ is non-degenerate for each $a \in S(\mathbb{C})$. By [4, Thm. 1.4 and Thm. 4.1], the cohomology group $\mathcal{H}^i_{\mathrm{dR}}(U/S, f_a)$ vanishes if $i \neq n+m-1$.

Now we consider the $(G, \widetilde{\chi}\times\rho)$-isotypic component of the connection $\mathcal{H}^{n+m-1}_{\mathrm{dR}}(U/S, f)$, which can be identified with $(\mathcal{H}^0\mathrm{pr}_{z+}\mathcal{E}^f)^{(G,\widetilde{\chi}\times\rho)}$. It remains to prove that the cohomology class defined by the differential form

$$\omega = \prod_{i=2}^{n} x_i^{a_i} \cdot \prod_{j=1}^{m} y_j^{-b_j} \frac{\mathrm{d}x_2}{x_2} \cdots \frac{\mathrm{d}x_n}{x_n} \frac{\mathrm{d}y_1}{y_1} \cdots \frac{\mathrm{d}y_m}{y_m}$$

is a cyclic vector for $\mathcal{H}^{n+m-1}_{\mathrm{dR}}(U/S, f)^{(G,\widetilde{\chi}\times\rho)}$.

Lemma 2.4.3. *Let $t_2, \cdots, t_n, s_1, \cdots, s_m$ be integers and set*

$$\widetilde{\omega} := \prod_{i=2}^{n} x_i^{t_i} \cdot \prod_{j=1}^{m} y_j^{s_j} \frac{\mathrm{d}x_2}{x_2} \cdots \frac{\mathrm{d}x_n}{x_n} \frac{\mathrm{d}y_1}{y_1} \cdots \frac{\mathrm{d}y_m}{y_m}$$

as a class in $\mathcal{H}^{n+m-1}_{\mathrm{dR}}(U/S, f)$. For each i and j such that $2 \le i \le n$ and $1 \le j \le m$ respectively, we have

$$(D - t_i/d)\widetilde{\omega} = x_i^d \cdot \widetilde{\omega} \quad \textit{and} \quad (D + s_j/d)\widetilde{\omega} = y_j^d \cdot \widetilde{\omega}.$$

Proof. We prove the identity for $(D - t_2)\widetilde{\omega}$. And the proofs for the rest are identical. By (2.4.2.1), we have $D\,\widetilde{\omega} = \frac{z\cdot\prod_j y_j^d}{\prod_i x_i^d}\,\widetilde{\omega}$. Then, by the definition of the relative twisted de Rham cohomology

$$\begin{aligned} 0 = \nabla_{U/S}\left(\prod_{i=2}^{n} x_i^{t_i} \cdot \prod y_j^{s_j} \frac{\mathrm{d}x_3}{x_3} \cdots \frac{\mathrm{d}x_n}{x_n} \frac{\mathrm{d}y_1}{y_1} \cdots \frac{\mathrm{d}y_m}{y_m}\right) &= t_2 \cdot \widetilde{\omega} + x_2 \cdot \partial_{x_2} f \cdot \widetilde{\omega} \\ = t_2 \cdot \widetilde{\omega} + x_2 \cdot \left(dx_2^{d-1} - dx_2^{-1}\frac{z\prod y_j^d}{\prod x_i^d}\right)\widetilde{\omega} &= d(x_2^d - (D - t_2/d))\,\widetilde{\omega}. \end{aligned}$$

This is exactly what we want to prove. ∎

We show that ω satisfies the hypergeometric differential equation $\mathrm{Hyp}(\alpha;\beta)$. By Lemma 2.4.3, we have

$$\prod_{i=2}^{n}(D-\alpha_i)\omega=\prod_{i=2}^{n}x_i^d\cdot\omega \quad\text{and}\quad \prod_{j=1}^{m}(D-\beta_j)\omega=\prod_{j=1}^{m}y_j^d\cdot\omega.$$

Then, we deduce from (2.4.2.1) that

$$\prod_{i=1}^{n}(D-\alpha_i)\omega=D\Big(\prod_{i=2}^{n}x_i^d\cdot\omega\Big)=z\prod_{j=1}^{m}y_j^d\cdot\omega=z\prod_{j=1}^{m}(D-\beta_j)\omega.$$

Using Lemma 2.4.3, we deduce that $\omega\neq 0$. So we get a nonzero morphism

$$\mathscr{D}_S/\mathrm{Hyp}(\alpha;\beta)\to\bigoplus_{i=0}^{n-1}\mathcal{O}_{\mathbb{G}_m}\cdot D^i\omega\subset\mathcal{H}^{n+m-1}_{\mathrm{dR}}(U/S,f)^{G,\widetilde{\chi}\times\rho} \tag{2.4.3.1}$$

defined by sending 1 to ω. Since the left-hand side is irreducible, and both sides have the same ranks, the above morphism is an isomorphism. By Proposition 2.3.1, ω is a cyclic vector of $\mathcal{H}yp(\alpha;\beta)\simeq\mathcal{H}^{n+m-1}_{\mathrm{dR}}(U/S,f)^{G,\widetilde{\chi}\times\rho}$. ◻

Remark 2.4.4. If we replace $\mathcal{E}^f$ by $(\mathbb{G}_m,\mathrm{d}-\mathrm{d}f)=(\mathbb{G}_m,\mathrm{d}+\mathrm{d}f)^\vee$, the direct sum $\bigoplus_{i=0}^{n-1}\mathcal{O}D^i\omega$ is the $(G,\widetilde{\chi}\times\rho)$-isotypic component of $\mathcal{H}^{n+m-1}_{\mathrm{dR}}(\mathbb{G}_m^{n+m}/\mathbb{G}_m,-f)$, isomorphic to the connection $\mathcal{H}yp_{(-1)^{n-m}}(\alpha;\beta)$. To see this, it suffices to notice that the corresponding identities in Lemma 2.4.3 become

$$(D-t_i/d)\omega_{t,s}=-x_i^d\omega_{t,s}\text{ and }(D+s_j/d)\omega_{t,s}=-y_j^d\omega_{t,s}$$

in this case. The rest of the proof relies on the above calculation and Remark 2.3.4.

2.4.5. Resonant case. When (α,β) is resonant, the modified hypergeometric $\mathscr{D}$-module $\mathcal{H}yp(*;\alpha;\beta)$ depends only on the classes of α and β modulo $\mathbb{Z}$. In [23, 6.3.8], Katz asked whether $\mathcal{H}yp(*;\alpha;\beta)$ is isomorphic to the connection $\mathcal{H}yp\big((\alpha_i+r_i);(\beta_j+s_j)\big)$ (2.1.1.1) for suitable integers $r_i,s_j\in\mathbb{Z}$. We provide a positive answer to this question in the following proposition.

Proposition 2.4.6. *When (α,β) is resonant, there exists a positive integer h depending on $\alpha \bmod \mathbb{Z}$ and $\beta \bmod \mathbb{Z}$, such that for any integers $r,s>h$, the modified hypergeometric $\mathscr{D}$-module $\mathcal{H}yp(*;\alpha;\beta)|_S$ is isomorphic to the hypergeometric connection $\mathcal{H}yp\big((\alpha_1,\alpha_2-r,\dots,\alpha_n-r);\beta+s\big)$.*

Proof. We may assume that $\alpha_1=0$. Let $\widetilde{\omega}_1,\dots,\widetilde{\omega}_n$ be a representative of a basis of the connection $\mathcal{H}^{n+m-1}_{\mathrm{dR}}(U/S,f)^{G,\widetilde{\chi}\times\rho}$. More precisely, we can write

$$\widetilde{\omega}_k=\sum_{e\in\mathbb{Z}^{n-1},f\in\mathbb{Z}^m}\epsilon_{k,e,f}\prod_{i=2}^{n}x_i^{a_i+d\cdot e_i}\prod_{j=1}^{m}y_j^{-b_j+d\cdot f_j}\frac{\mathrm{d}x_2}{x_2}\cdots\frac{\mathrm{d}x_n}{x_n}\frac{\mathrm{d}y_1}{y_1}\cdots\frac{\mathrm{d}y_m}{y_m},$$

where only finitely many $\epsilon_{k,e,f}$ are non-zero. We equip $\mathbb{Z}^{n+m-1}$ with the partial order defined by the relation that $a\geq b$ if $a-b\in\mathbb{N}^{n+m-1}$. Let (e_0,f_0) be a maximal element in the set $\{(e',f')\mid(e',f')\leq(e,f)\text{ if }\epsilon_{k,e,f}\neq 0\}$. Then we take h to be the the maximal value among $\{|(e_0)|_i,|(f_0)|_j\}$.

For any $r,s>h$, as in Proposition 2.4.1, we define a morphism of $\mathscr{D}$-modules:

$$\mathscr{D}_S/\mathrm{Hyp}(0,\alpha_2-r,\dots,\alpha_n-r;\beta+s)\to\bigoplus_{i=0}^{n-1}\mathcal{O}_{\mathbb{G}_m}\cdot D^i\omega\subset\mathcal{H}^{n+m-1}_{\mathrm{dR}}(U/S,f)^{G,\widetilde{\chi}\times\rho} \tag{2.4.6.1}$$

by sending 1 to

$$\omega = \prod_{i=2}^{n} x_i^{a_i - d\cdot r} \cdot \prod_{j=1}^{m} y_j^{-b_j - d\cdot s} \frac{\mathrm{d}x_2}{x_2} \cdots \frac{\mathrm{d}x_n}{x_n} \frac{\mathrm{d}y_1}{y_1} \cdots \frac{\mathrm{d}y_m}{y_m}.$$

Since for all (e, f) with $\epsilon_{k,e,f} \neq 0$, we have $a_i + d \cdot e_i \geq a_i - d \cdot r_i$ and $b_j + d \cdot f_j \geq b_j - d \cdot s_j$ for any i and j, we deduce that the class defined by $\prod_{i=2}^{n} x_i^{a_i + d\cdot e_i} \prod_{j=1}^{m} y_j^{-b_j + d\cdot f_j} \frac{\mathrm{d}x_2}{x_2} \cdots \frac{\mathrm{d}x_n}{x_n} \frac{\mathrm{d}y_1}{y_1} \cdots \frac{\mathrm{d}y_m}{y_m}$ lies in the image of (2.4.6.1) by Lemma 2.4.3. This morphism is surjective and, therefore, an isomorphism. □

3 Irregular Hodge filtration of hypergeometric connections

This section aims to calculate the (irregular) Hodge filtrations of hypergeometric connections (see Theorem 3.3.3 and Theorem 3.3.1).

In this section, let $n \geq m \geq 0$ be two integers, $\alpha = (\alpha_1, \dots, \alpha_n)$ and $\beta = (\beta_1, \dots, \beta_j)$ two sequences of non-decreasing rational numbers in $[0, 1)$.

3.1 Exponential mixed Hodge structures

To explain certain duality on the irregular Hodge filtration of hypergeometric connections, we use the language of exponential mixed Hodge structures introduced by Kontsevich-Soibelman [24]. We recall the basic definitions of exponential mixed Hodge structures from [18, Appx.].

Let X be a smooth algebraic variety and K a number field. We denote by $\mathrm{MHM}(X, K)$ the abelian category of *mixed Hodge modules* on X with coefficients in K. In particular, when $X = \mathrm{Spec}(\mathbb{C})$, the category $\mathrm{MHM}(X, K)$ is equivalent to the category of mixed K-Hodge structures. Moreover, the bounded derived categories $\mathrm{D}^b(\mathrm{MHM}(X, K))$ admit the six functor formalism. For more details about mixed Hodge modules, see [36].

Let $\pi\colon \mathbb{A}^1 \to \mathrm{Spec}(\mathbb{C})$ be the structure morphism. The category EMHS(K) of *exponential mixed Hodge structures* with coefficients in K is defined as the full subcategory of $\mathrm{MHM}(\mathbb{A}^1, K)$, whose objects N^{H} have vanishing cohomology on $\mathbb{A}^1$, i.e., satisfying $\pi_* N^{\mathrm{H}} = 0$.

There is an exact functor $\Pi\colon \mathrm{MHM}(\mathbb{A}^1, K) \to \mathrm{MHM}(\mathbb{A}^1, K)$ defined by

$$N^{\mathrm{H}} \mapsto s_*(N^{\mathrm{H}} \boxtimes j_! \mathcal{O}^{\mathrm{H}}_{\mathbb{G}_m}) \tag{3.1.0.1}$$

where $j\colon \mathbb{G}_{m,\mathbb{C}} \to \mathbb{A}^1$ is the inclusion and $s\colon \mathbb{A}^1 \times \mathbb{A}^1 \to \mathbb{A}^1$ is the summation map. The functor Π is a projector onto EMHS(K), i.e. it factors through EMHS(K) with essential image EMHS(K). Using this functor, the dual of an object M in EMHS(K) is defined by $\Pi([t \mapsto -t]^* \mathbb{D}(M))$, where t is the coordinate of $\mathbb{A}^1$.

For each object $\Pi(N^{\mathrm{H}})$ of the category EMHS(K), there exists a weight filtration $W^{\mathrm{EMHS}}_{\bullet}$ on $\Pi(N^{\mathrm{H}})$, defined by the weight filtration on N^{H}: $W^{\mathrm{EMHS}}_n \Pi(N^{\mathrm{H}}) := \Pi(W_n N^{\mathrm{H}})$. We will drop the superscript for simplicity.

The *de Rham fiber* functor from EMHS(K) to $\mathrm{Vect}_{\mathbb{C}}$ is defined by

$$\Pi(N^{\mathrm{H}}) \mapsto \mathrm{H}^1_{\mathrm{dR}}(\mathbb{A}^1, \Pi(N) \otimes \mathcal{E}^t), \tag{3.1.0.2}$$

where $\Pi(N)$ is the underlying $\mathscr{D}$-module of $\Pi(N^{\mathrm{H}})$ and $\mathcal{E}^t$ denotes the exponential $\mathscr{D}$-module $(\mathcal{O}_{\mathbb{A}^1}, \mathrm{d} + \mathrm{d}t)$.

The de Rham fiber functor is faithful and one can associate an *irregular Hodge filtration* $F^{\bullet}_{\mathrm{irr}}$ on the de Rham fibers of objects in EMHS(K) by [31, §6.b], compatible with the definitions in [15, 31, 33].

3.1.1 Objects of EMHS attached to regular functions

Let X be a smooth affine variety of dimension n and K a number field. We denote by K_X^{H} the trivial Hodge module on X with coefficients in K. For a regular function $g\colon X \to \mathbb{A}^1$ and an integer r, we consider the following exponential mixed Hodge structures

$$\mathrm{H}^r(X,g) := \Pi(\mathcal{H}^{r-n} g_* K_X^{\mathrm{H}}),\ \mathrm{H}_c^r(X,g) := \Pi(\mathcal{H}^{r-n} g_! K_X^{\mathrm{H}}).$$

The exponential mixed Hodge structures $\mathrm{H}^r(X,g)$, $\mathrm{H}_c^r(X,g)$ are mixed of weights at least r and mixed of weights at most r respectively by [18, A.19].

The de Rham fiber of $\mathrm{H}_?^r(X,g)$ is isomorphic to $\mathrm{H}_{\mathrm{dR},?}^r(X,g)$, and the irregular Hodge filtration on the de Rham fibers are identified with those on twisted de Rham cohomologies [42].

3.1.2 Irregular Hodge filtration and Newton monomial filtration

We briefly recall the definition of the irregular Hodge filtration on the twisted de Rham cohomology following [42]. Let X and g be as above, $j\colon X \to \bar{X}$ a smooth compactification of X, and $D := \bar{X}\backslash X$ the boundary divisor. The pair $(\bar{X}, D)$ is called a *good compactification* of the pair (X,g) if D is normal crossing and g extends to a morphism $\bar{g}\colon \bar{X} \to \mathbb{P}^1$.

Let P be the pole divisor of g. The twisted de Rham complex $(\Omega_{\bar{X}}^{\bullet}(*D), \nabla = \mathrm{d} + \mathrm{d}g)$ admits a decreasing filtration $F^{\lambda}(\nabla) := F^0(\lambda)^{\geq \lceil \lambda \rceil}$, indexed by non-negative real numbers λ, where $F^0(\lambda)$ is the Yu complex

$$\mathcal{O}_{\bar{X}}(\lfloor -\lambda P \rfloor) \xrightarrow{\nabla} \Omega_{\bar{X}}^1(\log D)(\lfloor (1-\lambda)P \rfloor) \to \cdots \to \Omega_{\bar{X}}^p(\log D)(\lfloor (p-\lambda)P \rfloor) \to \cdots .$$

The *irregular Hodge filtration* on the de Rham cohomology $\mathrm{H}_{\mathrm{dR}}^1(X,g)$ is defined by

$$F_{\mathrm{irr}}^{\lambda}\mathrm{H}_{\mathrm{dR}}^i(X,g) := \mathrm{im}(\mathbb{H}^i(\bar{X}, F^{\lambda}(\nabla)) \to \mathrm{H}_{\mathrm{dR}}^i(X,g)), \tag{3.1.0.3}$$

which is independent of the choice of the good compactification $(\bar{X}, D)$ [42, Thm. 1.7].

When X is isomorphic to a torus $\mathbb{G}_m^n$, the regular function g on X is a Laurent polynomial of the form $\sum_{P=(p_1,\ldots,p_n)} c(P)x^P$. We refine the normal fan of the Newton polytope $\Delta(g)$ to make the associated toric variety X_{tor} smooth proper. Although $(X_{\mathrm{tor}}, D_{\mathrm{tor}} = X_{\mathrm{tor}}\backslash X)$ is not a good compactification for the pair (X,g) in general, we can still define $F_{\mathrm{NP}}^{\lambda}(\nabla)$ and the *Newton polyhedron filtration* $F_{\mathrm{NP}}^{\lambda}\mathrm{H}_{\mathrm{dR}}^1(U,\nabla)$ similarly to that in (3.1.0.3) by replacing the good compactification $(\bar{X}, D)$ with $(X_{\mathrm{tor}}, D_{\mathrm{tor}})$,

When g is non-degenerate with respect to $\Delta(g)$, the only non-vanishing twisted de Rham cohomology group of the pair (X,g) is the middle cohomology group $\mathrm{H}_{\mathrm{dR}}^n(X,g)$ by [4, Thm 1.4], and the irregular Hodge filtration $F_{\mathrm{irr}}^{\bullet}$ agrees with the Newton polyhedron filtration $F_{\mathrm{NP}}^{\bullet}$ on $\mathrm{H}_{\mathrm{dR}}^n(X,g)$ [42, Thm. 4.6]. In particular, we have

$$\mathbb{H}^i(X_{\mathrm{tor}}, F_{\mathrm{NP}}^{\lambda}(\nabla)) = \mathrm{H}^i(\Gamma(X_{\mathrm{tor}}, F_{\mathrm{NP}}^{\lambda}(\nabla))),$$

which allows us to compute the irregular Hodge filtration by knowledge of $\Delta(g)$.

Now, we present an explicit way to calculate the Newton polyhedron filtration. For a cohomology class $\omega = x^Q \frac{\mathrm{d}x_1}{x_1} \wedge \cdots \wedge \frac{\mathrm{d}x_n}{x_n}$ such that the lattice point $Q = (q_1, \ldots, q_n)$ lies in $\mathbb{R}_{\geq 0}\Delta(g)$, we define $w(Q)$ to be the weight of Q in the sense of [4], i.e. the minimal positive real number w such that $Q \in w \cdot \Delta(g)$. The associated cohomology class of ω lies in $F_{\mathrm{NP}}^{\lambda}\mathrm{H}_{\mathrm{dR}}^n(X,g)$ if

$$\omega \in \Gamma(X_{\mathrm{tor}}, \Omega_{X_{\mathrm{tor}}}^n(\log D_{\mathrm{tor}})(\lfloor (n-\lambda)P \rfloor)).$$

Notice that each ray ρ in the normal fan of $\Delta(g)$ corresponds to an irreducible component P_ρ of P. Let v_ρ be a primitive vector of the ray ρ. Then, the multiplicity of ω along P_ρ is given by $< Q, v_\rho >$ [19, p. 61]. Taking the multiplicities of P_ρ in P into account, we have

$$x^Q \frac{\mathrm{d}x_1}{x_1} \wedge \cdots \wedge \frac{\mathrm{d}x_n}{x_n} \in F_{\mathrm{NP}}^{n-w(Q)} \mathrm{H}^n_{\mathrm{dR}}(X, g). \tag{3.1.0.4}$$

as remarked in [42, p. 126 footnote].

3.1.3 The EMHS associated with hypergeometric connections

In this subsection, we assume $\alpha_1 = 0$ and let $\widetilde{\chi} \times \rho$ be the product of characters associated with α_i and β_j in (2.3.0.1).

Definition 3.1.1. Let K be the number field $\mathbb{Q}(\zeta_d^{a_i}, \zeta_d^{b_j})$ and $a \in S(\mathbb{C})$. For $? \in \{\varnothing, c\}$, we define

$$E_?(a;\alpha;\beta) := \mathrm{H}^{n+m-1}(\mathbb{G}_m^{n+m-1}, f_a)^{(G, \widetilde{\chi \times \rho})}$$

as exponential mixed Hodge structures with coefficients in K in the sense of (3.1.1).

By Proposition 2.3.1 and the base change theorem, the de Rham fiber of $E(a;\alpha;\beta)$ is isomorphic to the fiber of $\mathcal{H}yp_\lambda(\alpha;\beta)$ at $a \cdot \lambda \in S(\mathbb{C})$, for $\lambda \in \mathbb{Q}^\times$.

Let t be the largest natural number such that $\alpha_t = 0$. We let $\bar{\alpha}$ and $\bar{\beta}$ be the sequences of rational numbers defined by

$$\bar{\alpha}_i = \begin{cases} 0 & 1 \le k \le t, \\ 1 - \alpha_{n+t+1-k} & t+1 \le k \le n, \end{cases} \quad \text{and} \quad \bar{\beta}_k = 1 - \beta_k. \tag{3.1.1.1}$$

Proposition 3.1.2. *(1). The dual of the exponential mixed Hodge structure $E_c(a;\alpha;\beta)$ is isomorphic to $E((-1)^{n-m}a;\bar{\alpha};\bar{\beta})$.*

(2). When (α,β) is non-resonant, the exponential mixed Hodge structures $E_?(a;\alpha;\beta)$ for $? \in \{\varnothing, c\}$ are all isomorphic. In particular, they are pure of weight $n+m-1$.

Proof. (1) The EMHS $\mathrm{H}_c^{n+m-1}(\mathbb{G}_m^{n+m-1}, f_a)$ is dual to $\mathrm{H}^{n+m-1}(\mathbb{G}_m^{n+m-1}, -f_a)$, which is also isomorphic to $\mathrm{H}^{n+m-1}(\mathbb{G}_m^{n+m-1}, f_{(-1)^{n-m}a})$. We deduce the first assertion by taking their corresponding isotypic components.

(2) Since the de Rham fiber functor is faithful, the forget support morphism

$$E_c(a;\alpha;\beta) \to E(a;\alpha;\beta)$$

is an isomorphism by Corollary 2.3.3. Hence, the exponential mixed Hodge structures $E_c(a;\alpha;\beta)$ and $E(a;\alpha;\beta)$ are isomorphic, and are pure of weight $n+m-1$. □

Remark 3.1.3. We can define confluent hypergeometric motives as exponential motives in the sense of [17], such that the exponential mixed Hodge structures $E(1;\alpha;\beta)$ and the hypergeometric connections $\mathcal{H}yp(\alpha,\beta)$ are their Hodge realizations and $\mathscr{D}$-module realizations respectively.

More precisely, assume $n > m$ and let ζ_{n-m} be an $(n-m)$-th primitive root of unity. The group μ_d^{n+m-1} acts on $\mathbb{G}^{n+m-1}_{m,(x_i,y_j)} \times \mathbb{G}_{m,t}$ as before on the coordinates (x_i, y_j), and the group μ_{n-m} acts on the coordinates (x_i, y_j, t) by

$$\zeta_{n-m} \cdot (x_i, y_j, t) = (\zeta_{n-m}^{-1} x_i, \zeta_{n-m}^{-1} y_j, \zeta_{n-m} t).$$

Then we define the confluent hypergeometric motives as

$$\mathrm{H}^{n+m-1}(\mathbb{G}_m^{n+m-1}, f_1)^{(\mu_d^{n+m-1}\times\mu_{n-m},\widetilde{\chi}\times\rho\times 1)},$$

which is a priori defined over $K(\zeta_{n-m})$ with coefficients in K.

Here, the field of definition $K(\zeta_{n-m})$ is not optimal. For example, using an argument similar to that in [29, Rem. 3.3] and the Galois descent [17, Thm. 5.2.4], one can show that these motives are defined over K. If L is a subfield of K such that $\mathrm{Gal}(\mathbb{C}/L)$ preserves both the sets $\{\exp(2\pi i\alpha_i)\}$ and $\{\exp(2\pi i\beta_j)\}$, then these motives can further descend to L (see also [5, Thm. 1.1] for a related discussion).

3.2 A basis in relative twisted de Rham cohomology

In this subsection, we assume $\alpha_1 = 0$. We define positive integers $s_1, \dots, s_{m+1}$ by

$$s_r = \begin{cases} 1 & r = 0 \\ \#\{i\colon \alpha_i < \beta_r\} & 1 \le r \le m \\ n+1 & r = m+1 \end{cases}$$

and for r and ℓ such that $0 \le r \le m$ and $1 \le \ell \le s_{r+1} - s_r$, we set

$$g_{r,\ell} = x_2^{a_2}\cdots x_{s_r+\ell-1}^{a_{s_r+\ell-1}}\cdot x_{s_r+\ell}^{a_{s_r+\ell}-d}\cdots x_n^{a_n-d}\cdot y_1^{d-b_1}\cdots y_r^{d-b_r}\cdot y_{r+1}^{2d-b_{r+1}}\cdots y_m^{2d-b_m},$$

Let $\eta = \frac{\mathrm{d}x_2}{x_2}\cdots\frac{\mathrm{d}x_n}{x_n}\frac{\mathrm{d}y_1}{y_1}\cdots\frac{\mathrm{d}y_m}{y_m}$ and $\omega_{r,\ell} = g_{r,\ell}\cdot\eta$ be the corresponding differential forms in $\mathcal{H}_{\mathrm{dR}}^{n+m-1}(U/S, \pm f)^{(G,\widetilde{\chi}\times\rho)}$, where U and S are defined in (2.3.0.3).

Proposition 3.2.1. *If (α,β) is non-resonant, the cohomology classes defined by*

$$\omega_{r,\ell}, \quad 0 \le r \le m, \quad 1 \le \ell \le s_{r+1} - s_r$$

in $\mathcal{H}_{\mathrm{dR}}^{n+m-1}(U/S, \pm f)^{(G,\widetilde{\chi}\times\rho)}$ form a basis over $\mathcal{O}_S$.

Proof. It suffices to show that $\mathrm{span}(\omega_{r,\ell}) = \mathrm{span}(D^i\omega \mid 0 \le i \le n-1)$ for a cyclic vector ω.

To a Laurent monomial $g = \prod x_i^{u_i}\prod y_j^{v_j}$ in variables $\{x_i\}_{i=2}^n$ and $\{y_j\}_{j=1}^m$, we associate a lattice point $\mathcal{P}(g) = (u_2, \dots, u_n, v_1, \dots, v_m) \in \mathbb{Z}^{n+m-1} \subset \mathbb{R}^{n+m-1}$. If $\omega = g\cdot\eta$ is the product of a monomial g with the differential form η, we set $\mathcal{P}(\omega) := \mathcal{P}(g)$ for the corresponding point.

Let π_1 and π_2 be the projections from $\mathbb{R}^{n+m-1}$ to $\mathbb{R}^{n-1}_{u_i}$ and $\mathbb{R}^m_{v_j}$ respectively. The for the differential forms $\omega_{r,\ell}$, we have

$$\pi_1(\mathcal{P}(\omega_{r,\ell})) = (a_2, \dots, a_{s_r+\ell-1}, a_{s_r+\ell} - d, \dots, a_n - d)$$

and

$$\pi_2(\mathcal{P}(\omega_{r,\ell})) = (d - b_1, \dots, d - b_r, 2d - b_{r+1}, \dots, 2d - b_m).$$

It follows from Lemmas 2.2.2, 2.2.3, and 2.2.4 that the fibers of all $\omega_{r,\ell}$ at $a \in S(\mathbb{C})$ lie in the cone $\mathbb{R}_{\ge 0}\cdot\Delta(f_a)$

Let P_i and Q_j be the points corresponding to monomials x_i^d and y_j^d respectively for $2 \le i \le n$ and $1 \le j \le m$.

Lemma 3.2.2. *For a point $P \in \mathbb{Z}^{n+m-1}$ and two integers $2 \le i_0 \le n$ and $1 \le j_0 \le m$, let ω_0, ω_1 and ω_2 be the corresponding differential forms of the points $P, P+Q_{j_0}$ and $P+P_{i_0}$ in $\mathbb{Z}^{n+m-1}$. If the i_0-th coordinate of P is different from the negative of the j_0-th coordinate of P, then we have*

$$\mathrm{span}(\omega_0, \omega_2) = \mathrm{span}(\omega_1, \omega_2) \quad \text{in} \quad \mathrm{H}_{\mathrm{dR}}^{n+m-1}(U/S, f)^{(G,\widetilde{\chi}\times\rho)}.$$

Proof. Let P be the point $(t_i, s_j) \in \mathbb{Z}^{n+m-1}$ and ω_0 be the associated differential form. By the assumption, we have $t_i \neq -s_j$. Then

$$\operatorname{span}(\omega_0, (D - t_{i_0})\omega_0) = \operatorname{span}((D - t_{i_0})\omega_0, (D + s_{j_0})\omega_0).$$

At last, notice that we have $\omega_1 = (D + s_{j_0})\omega_0$ and $\omega_2 = (D - t_{i_0})\omega_0$ by Lemma 2.4.3 and Remark 2.4.4. ☐

Step 1: If $s_1 - s_0 = 0$, we skip this step and put $\omega_{r,\ell}^{(1)} = \omega_{r,\ell}$ for any r, ℓ. Otherwise, for $r = 0$ and $1 \leq \ell \leq s_1 - s_0$, we replace the differential forms $\omega_{0,\ell}$ by differential forms $\omega_{0,\ell}^{(1)}$ of the forms $g \cdot \eta$ for some monomials g, such that

$$\mathcal{P}(\omega_{0,\ell}^{(1)}) = \mathcal{P}(\omega_{0,\ell}) - Q_1.$$

More precisely, we keep the first $n-1$ coordinates of $\mathcal{P}(\omega_{0,\ell})$ unchanged and replace the last m coordinates of $\mathcal{P}(\omega_{0,\ell})$ by that of $\mathcal{P}(\omega_{0,\ell}^{(1)})$:

$$(d - b_1, 2d - b_2, \ldots, 2d - b_m).$$

In particular, by Lemma 2.4.3, one has

$$(D + 1 - \beta_1)\omega_{0,\ell}^{(1)} = \omega_{0,\ell}, \quad (D - \alpha_{\ell+1})\omega_{0,\ell}^{(1)} = \omega_{0,\ell+1},$$

and

$$(D - \alpha_{s_1 - s_0})\omega_{0,s_1}^{(1)} = \omega_{e,1},$$

where e is the least integer such that $s_e > s_0 = 1$.

We also put $\omega_{r,\ell}^{(1)} = \omega_{r,\ell}$ for $r \geq 1$. Then use Lemma 3.2.2 for $\omega_0 = \omega_{0,s_1-s_0}^{(1)}$, $\omega_1 = \omega_{0,s_1-s_0}$, and $\omega_2 = \omega_{e,1}$, we have

$$\begin{aligned}\operatorname{Span}\{\omega_{r,\ell} \mid r, \ell\} &= \operatorname{Span}\Big(\ldots, \omega_{0,s_1-s_0}(= (D + 1 - \beta_1)\omega_{0,s_1-s_0}^{(1)}), \omega_{e,1}(= (D - \alpha_{s_0})\omega_{0,s_1-s_0}^{(1)}), \ldots\Big) \\ &= \operatorname{Span}(\{\omega_{0,1}, \ldots, \omega_{0,s_1-s_0-1}, \omega_{0,s_1-s_0}^{(1)}\} \cup \{\omega_{r,\ell}^{(1)} \mid r \geq 1, \ell\})\end{aligned}$$

where $0 \leq r \leq m$ and $1 \leq \ell \leq s_{r+1} - s_r$. Continue using Lemma 3.2.2 for $\omega_0 = \omega_{0,\ell}^{(1)}$, $\omega_1 = \omega_{0,\ell}$, and $\omega_2 = \omega_{0,\ell+1}$ for $\ell = s_1 - s_0 - 1, \ldots, s_1 - 1$, we have

$$\begin{aligned}\operatorname{Span}\{\omega_{r,\ell} \mid r, \ell\} &= \operatorname{Span}(\{\omega_{0,1}, \ldots, \omega_{0,s_1-s_0-1}, \omega_{0,s_1-s_0}^{(1)}\} \cup \{\omega_{r,\ell}^{(1)} \mid r \geq 1, \ell\}) \\ &= \operatorname{Span}(\{\omega_{0,1}, \omega_{0,2}^{(1)} \ldots, \omega_{0,s_1-s_0}^{(1)}\} \cup \{\omega_{r,\ell}^{(1)} \mid r \geq 1, \ell\}) \\ &= \operatorname{Span}(\omega_{r,\ell}^{(1)} \mid r, \ell).\end{aligned}$$

Step $i \geq 2$: Assume that we have already obtained elements $\omega_{r,\ell}^{(i-1)}$ for $i \geq 2$. If $s_i = s_{i-1}$, we skip this step and put $\omega_{r,\ell}^{(i)} = \omega_{r,\ell}^{(i-1)}$ for any r and ℓ. Otherwise, let $\omega_{r,\ell}^{(i)}$ be differential forms of the forms $g \cdot \eta$ for some monomials g, such that

$$\mathcal{P}(\omega_{r,\ell}^{(i)}) = \begin{cases} \mathcal{P}(\omega_{r,\ell}^{(i-1)}) - Q_i & \text{if } r \leq i-1, \\ \mathcal{P}(\omega_{r,\ell}^{(i-1)}) & \text{if } i \leq r \leq m. \end{cases}$$

More precisely, when $r \leq i-1$, we keep the first $n-1$ coordinates of $\mathcal{P}(\omega_{r,\ell}^{(i-1)})$ unchanged, and replace the last m coordinates of $\mathcal{P}(\omega_{r,\ell}^{(i-1)})$ by that of $\mathcal{P}(\omega_{r,\ell}^{(i)})$:

$$(d-b_1,\dots,d-b_i,2d-b_{i+1},\dots,2d-b_m).$$

Similar to step 1, we use Lemma 2.4.3 and Lemma 3.2.2 $(s_{r+1}-s_r)$-many times to deduce

$$\mathrm{Span}(\omega_{r,\ell}^{(i)}\mid r,\ell)=\mathrm{Span}(\omega_{r,\ell}^{(i-1)}\mid r,\ell)=\mathrm{Span}(\omega_{r,\ell}\mid r,\ell),$$

where $0\leq r\leq m$, and $1\leq \ell\leq s_{r+1}-s_r$.

After Step m: After m steps, we get $\omega_{r,\ell}^{(m)}$ such that

$$\mathcal{P}(\omega_{r,\ell}^{(m)})=(a_2,\dots,a_{s_r+\ell-1},a_{s_r+\ell}-d,\dots,a_n-d,d-b_1,\dots,d-b_m).$$

Note that the there is a bijection between $\{(r,\ell)\}_{0\leq r\leq m,1\leq \ell\leq s_{r+1}-s_r}$ and $\{1,\cdots,n\}$ by sending (r,ℓ) to $s_r+\ell-1$. We set $\widetilde{\omega}_{s_r+\ell-1}=\omega_{r,\ell}^{(m)}$ via this map. Then by Lemma 2.4.3, we have $\widetilde{\omega}_{i+1}=(D-1+\frac{a_{i+1}}{d})\widetilde{\omega}_i$ for $1\leq i\leq n-1$. It follows that

$$\begin{aligned}\mathrm{Span}(D^i\widetilde{\omega}_1\mid 0\leq i\leq n-1) &= \mathrm{Span}(\widetilde{\omega}_i\mid 1\leq i\leq n)\\ &= \mathrm{Span}(\omega_{r,\ell}^{(m)}\mid r,\ell)\\ &= \mathrm{Span}(\omega_{r,\ell}\mid r,\ell).\end{aligned}$$

By Proposition 2.4.1 and Remark 2.4.2, $\widetilde{\omega}_1$ is a cyclic vector, from which we showed that $\{\omega_{r,\ell}\}_{r,\ell}$ form a basis. This finishes the proof. □

3.3 Calculation of the irregular Hodge filtration

Recall that a non-resonant hypergeometric connection is rigid. Hence, it underlies an irregular mixed Hodge module on $\mathbb{P}^1$ [32, Thm. 0.7], and therefore, admits a unique irregular Hodge filtration $F_{\mathrm{irr}}^{\bullet}$ (up to a shift). When $n=m$, the irregular mixed Hodge module structure coincides with the variation of Hodge structures on $\mathcal{H}yp(\alpha,\beta)$.

Recall that for (α,β), we defined in (1.0.1.1) the numbers

$$\theta(k)=(n-m)\alpha_k+\#\{i\mid \beta_i<\alpha_k\}+(n-k)-\sum_{i=1}^{n}\alpha_i+\sum_{j=1}^{m}\beta_j.$$

Theorem 3.3.1. *Assume (α,β) is non-resonant.*

(1). When $\alpha_1=0$, via the isomorphism $\mathcal{H}yp(\alpha,\beta)\simeq\mathcal{H}_{\mathrm{dR}}^{n+m-1}(U/S,f)^{(G,\widetilde{\chi}\times\rho)}$, the irregular Hodge filtration on $\mathcal{H}yp(\alpha,\beta)$ can be identified with the following filtration of sub-bundles:

$$F_{\mathrm{irr}}^{p}\mathcal{H}_{\mathrm{dR}}^{n+m-1}(U/S,f)^{(G,\widetilde{\chi}\times\rho)}=\bigoplus_{n+m-1-w(\omega_{r,s})\geq p}\omega_{r,s}\mathcal{O}_S.$$

(2). Up to an $\mathbb{R}$-shift, the jumps of the irregular Hodge filtration on $\mathcal{H}yp(\alpha,\beta)$ occur at $\theta(k)$ and for any $p\in\mathbb{R}$ we have

$$\mathrm{rk}\,\mathrm{gr}_{F_{\mathrm{irr}}}^{p}\mathcal{H}yp(\alpha;\beta)=\#\theta^{-1}(p).$$

Remark 3.3.2. (i) By [33, Rem. 6.3], the irregular Hodge filtration satisfies the Griffiths' transversality, that is, $\nabla(F^p_{\mathrm{irr}}\mathcal{H}yp(\alpha,\beta)) \subset \Omega^1_S \otimes F^{p-1}_{\mathrm{irr}}\mathcal{H}yp(\alpha,\beta)$, for all $p \in \mathbb{R}$.

(ii) Inspired by the Griffiths' transversality, we expect that there exist oper structures on the hypergeometric connections, which refine the irregular Hodge filtrations. An oper structure is essential in the geometric Langlands correspondence [6, 43, 21].

To prove the above theorem, we study the Hodge numbers of the irregular Hodge filtration on fibers as explained in Section 1.2.

Theorem 3.3.3. *Up to an $\mathbb{R}$-shift, the jumps of the irregular Hodge filtration $F^\bullet_{\mathrm{irr}}$ on the fiber $\mathcal{H}yp(\alpha;\beta)_a$ occur at $\theta(k)$ for $1 \le k \le n$. Moreover, we have $\dim \mathrm{gr}^p_{F_{\mathrm{irr}}}\mathcal{H}yp(\alpha;\beta)_a = \#\theta^{-1}(p)$ for any $p \in \mathbb{R}$.*

3.3.1 Proof of Theorem 3.3.3

Proof. We may assume $\alpha_1 = 0$ by (2.1.1.2). By Proposition 2.3.1 and Definition 3.1.1, we have

$$\begin{aligned} F^\bullet_{\mathrm{irr}}\mathcal{H}yp(\alpha;\beta)_a &\simeq F^\bullet_{\mathrm{irr}}\mathrm{H}^{n+m-1}_{\mathrm{dR}}(\mathbb{G}^{n+m-1}_m, f_a)^{(G,\widetilde{\chi}\times\rho)} \\ &\simeq F^\bullet_{\mathrm{irr}}\mathrm{H}^{n+m-1}_{\mathrm{dR}}(\mathbb{G}^{n+m-1}_m, -f_{(-1)^{n-m}a})^{(G,\widetilde{\chi}\times\rho)}, \end{aligned} \tag{3.3.3.1}$$

where $\widetilde{\chi}$ and ρ are products of characters corresponding to α_i and β_j from (2.3.0.1). So it suffices to compute the irregular Hodge filtration on the twisted de Rham cohomologies $\mathrm{H}^{n+m-1}_{\mathrm{dR}}(\mathbb{G}^{n+m}_m, \pm f_a)^{(G,\widetilde{\chi}\times\rho)}$. Since f_a is non-degenerate with respect to $\Delta(f_a)$, we can compute the filtration in terms of Newton polyhedron filtration.

Let $\omega_{r,\ell}$ be the basis of $\mathcal{H}yp(\alpha;\beta)_a$ from Proposition 3.2.1. Recall that $w(\omega_{r,\ell})$ is the minimal positive real number w such that $\mathcal{P}(g_{r,\ell}) \in w\cdot\Delta(f_a)$. It follows from (3.1.0.4) that

$$\omega_{r,\ell} \in F^{n+m-1-w(\omega_{r,\ell})}_{\mathrm{irr}}\mathrm{H}^{n+m-1}_{\mathrm{dR}}(\mathbb{G}^{n+m-1}_m, \pm f_a)^{(G,\widetilde{\chi}\times\rho)}.$$

We consider an auxiliary filtration $G^\bullet$ on $\mathrm{H}^{n+m-1}_{\mathrm{dR}}(\mathbb{G}^{n+m-1}_m, \pm f_a)^{(G,\widetilde{\chi}\times\rho)}$ defined by

$$G^p := \mathrm{span}\{\omega_{r,\ell} \mid n+m-1-w(\omega_{r,\ell}) \ge p\}. \tag{3.3.3.2}$$

By the following lemmas 3.3.4, 3.3.5, and 3.3.6, the filtration $F^\bullet$ coincides with $G^\bullet$, which finishes the proof of the theorem. □

Lemma 3.3.4. *We set $\theta(n+1) = \theta(1)$. For $0 \le r \le m$, $1 \le \ell \le s_{r+1} - s_r$, we have*

$$n+m-1-w(\omega_{r,\ell}) = \theta(s_r+\ell).$$

Lemma 3.3.5. *For $0 \le p \le n+m-1$, we have*

$$\dim \mathrm{gr}^p_G\mathrm{H}^{n+m-1}_{\mathrm{dR}}(\mathbb{G}^{n+m-1}_m, \pm f_a)^{(G,\widetilde{\chi}\times\rho)} = \dim \mathrm{gr}^{n+m-1-p}_G\mathrm{H}^{n+m-1}_{\mathrm{dR}}(\mathbb{G}^{n+m-1}_m, \mp f_a)^{(G,\widetilde{\chi}^{-1}\times\rho^{-1})}.$$

Lemma 3.3.6. *The two filtrations $F^\bullet_{\mathrm{irr}}$ and $G^\bullet$ coincide.*

Proof of Lemma 3.3.4. By Lemmas 2.2.2, 2.2.3, and 2.2.4, the weight $w(\omega_{r,\ell})$ equals to the number $\max_k\{h_k(g_{r,\ell})/d\}$, where h_k are defined in (2.2.2.1), (2.2.3.1), and (2.2.4.1). We can check that

$$w(\omega_{r,\ell}) = h_{s_r+\ell}(g_{r,\ell})/d,$$

where we put $h_1 = \ldots = h_n = h_{n+1}$ when $n = m$. Now it suffices to check that $n+m-1-w(\omega_{r,\ell})$ agrees with one of the jumps of the irregular Hodge numbers of $\mathcal{H}yp(\alpha;\beta)_a$.

If $s_r+\ell=n+1$, the monomial $g_{m,n+1-s_m}$ corresponds to the point

$$(a_2,\dots,a_n,d-b_1,\dots,d-b_m).$$

Then we have

$$n+m-1-h_{n+1}(g_{m,n+1-s_m})/d=n-1-\sum_{i=1}^{n}\alpha_i+\sum_{j=1}^{m}\beta_j=\theta(1).$$

If $s_r+\ell<n+1$, we have

$$\begin{aligned}&n+m-1-h_{s_r+\ell}(g_{r,\ell})/d\\=&n+m-1-\Big(\sum_{i=1}^{n}\alpha_i-(n+1-s_r-\ell)-\sum_{j=1}^{m}\beta_j+(2m-r)-(n-m)(\alpha_{s_r+\ell}-1)\Big)\\=&(n-m)\alpha_{s_r+\ell}+r+(n-s_r-\ell)-\sum_{i=1}^{n}\alpha_i+\sum_{j=1}^{m}\beta_j,\end{aligned}$$

which is exactly $\theta(s_r+\ell)$. □

Proof of Lemma 3.3.5. For simplicity, we write

$$\delta_p^{\pm}(\alpha,\beta):=\dim\mathrm{gr}_G^p\mathrm{H}_{\mathrm{dR}}^{n+m-1}(\mathbb{G}_m^{n+m-1},\pm f_a)^{(G,\widetilde{\chi}\times\rho)}. \tag{3.3.6.1}$$

Recall that in (3.1.1.1), we let t be the biggest natural number such that $\alpha_t=0$. For $1\le k\le t$, the numbers α_k and $\bar\alpha_{t+1-k}$ are 0. And for $t+1\le k\le n$, we have $\bar\alpha_{n-k+t+1}=1-\alpha_k$. Then

$$\sum_{i=1}^{n}\alpha_i+\sum_{i=1}^{n}\bar\alpha_i=n-t \text{ and } \sum_{j=1}^{m}\beta_j+\sum_{j=1}^{m}\bar\beta_j=m.$$

Similar to the number $\theta(k)$, we let $\bar\theta(k)$ be the numbers

$$(n-m)\bar\alpha_k+\#\{i\mid\bar\beta_i<\bar\alpha_k\}+(n-k)-\sum_{i=1}^{n}\bar\alpha_i+\sum_{j=1}^{m}\bar\beta_j,\ 1\le k\le n$$

for the sequences $\bar\alpha$ and $\bar\beta$. Then for $1\le k\le t$, we have

$$\begin{aligned}\theta(k)+\bar\theta(t+1-k)=&\Big(n-k-\sum_{i=1}^{n}\alpha_i+\sum_{j=1}^{m}\beta_j\Big)+\Big(n-(t+1-k)-\sum_{i=1}^{n}\bar\alpha_i+\sum_{j=1}^{m}\bar\beta_j\Big)\\=&(2n-t-1)-(n-t)+m=n+m-1.\end{aligned}$$

For $t+1\le k\le n$, we have

$$\begin{aligned}&\theta(k)+\bar\theta(n-k+t+1)\\=&\Big((n-m)\alpha_k+\#\{j\mid\beta_j<\alpha_k\}+n-k-\sum_{i=1}^{n}\alpha_i+\sum_{j=1}^{m}\beta_j\Big)\\+&\Big((n-m)\bar\alpha_{n-k+t+1}+\#\{j\mid\bar\beta_j<\bar\alpha_{n-k+t+1}\}+n-(n-k+t+1)-\sum_{i=1}^{n}\bar\alpha_i+\sum_{j=1}^{m}\bar\beta_j\Big)\\=&(n-m)+m+(n-t-1)-(n-t)+m=n+m-1.\end{aligned}$$

So there exists a permutation $\sigma \in S_n$ such that $\theta(k) + \bar{\theta}(\sigma(k)) = n + m - 1$. It follows that

$$\begin{aligned}\delta_p^{\pm}(\alpha,\beta) =& \#\{k \mid \theta(k) = p\} = \#\{k \mid n+m-1-p = n+m-1-\theta(k)\} \\ =& \#\{k \mid \bar{\theta}(k) = n+m-1-p\} = \delta_{n+m-1-p}^{\mp}(\bar{\alpha},\bar{\beta}).\end{aligned}$$

□

Proof of Lemma 3.3.6. For simplicity, we write

$$h_p^{\pm}(\alpha,\beta) := \dim \mathrm{gr}_{F_{\mathrm{irr}}}^p \mathrm{H}_{\mathrm{dR}}^{n+m-1}(\mathbb{G}_m^{n+m-1}, \pm f_a)^{(G,\widetilde{\chi}\times\rho)}. \tag{3.3.6.2}$$

By Lemma 3.3.4, for every $p \in \mathbb{Q}$, we have

$$G^p \mathrm{H}_{\mathrm{dR}}^{n+m-1}(\mathbb{G}_m^{n+m}, \pm f_a)^{(G,\widetilde{\chi}\times\rho)} \subset F_{\mathrm{irr}}^p \mathrm{H}_{\mathrm{dR}}^{n+m-1}(\mathbb{G}_m^{n+m}, \pm f_a)^{(G,\widetilde{\chi}\times\rho)}, \tag{3.3.6.3}$$

which implies that $\sum_{q\leq p} \delta_q^{\pm}(\alpha,\beta) \leq \sum_{q\leq p} h_q^{\pm}(\alpha,\beta)$.

To prove the reverse inclusion, we consider the duality between the two filtered vector spaces $(\mathrm{H}_{\mathrm{dR}}^{n+m-1}(\mathbb{G}_m^{n+m-1}, \pm f_a)^{(G,\widetilde{\chi}\times\rho)}, F_{\mathrm{irr}}^{\bullet})$ and $(\mathrm{H}_{\mathrm{dR}}^{n+m-1}(\mathbb{G}_m^{n+m-1}, \mp f_a)^{(G,\widetilde{\chi}^{-1}\times\rho^{-1})}, F_{\mathrm{irr}}^{\bullet})$, induced by Proposition 3.1.2 and [42, Thm. 2.2]. More precisely, we have

$$h_p^{\pm}(\alpha,\beta) = h_{n+m-1-p}^{\mp}(\bar{\alpha},\bar{\beta}). \tag{3.3.6.4}$$

Combining Lemma 3.3.5 and the equations (3.3.6.3) and (3.3.6.4), we see, for any $p \in \mathbb{R}$, that

$$\begin{aligned}&\dim G^p \mathrm{H}_{\mathrm{dR}}^{n+m-1}(\mathbb{G}_m^{n+m}, \pm f_a)^{(G,\widetilde{\chi}\times\rho)} \\ =& \sum_{q\leq p} \delta_q^{\pm}(\alpha,\beta) \leq \sum_{q\leq p} h_q^{\pm}(\alpha,\beta) = \sum_{q\geq n+m-1-p} h_q^{\mp}(\bar{\alpha},\bar{\beta}) \\ \leq& \sum_{q\geq n+m-1-p} \delta_q^{\mp}(\bar{\alpha},\bar{\beta}) = \sum_{q\leq p} \delta_q^{\pm}(\alpha,\beta) \\ =& \dim G^p \mathrm{H}_{\mathrm{dR}}^{n+m-1}(\mathbb{G}_m^{n+m}, \pm f_a)^{(G,\widetilde{\chi}\times\rho)}.\end{aligned}$$

Hence, both sides in (3.3.6.3) have the same dimension for every p. Then Lemma 3.3.6 follows. □

3.3.2 Proof of Theorem 3.3.1

Proof. We may assume $\alpha_1 = 0$ by (2.1.1.2). By [32, Prop. 3.54] and [28, Prop. 11.22], the irregular Hodge filtration on $\mathcal{H}yp(\alpha,\beta)$ induces those on fibers $\mathcal{H}yp(\alpha,\beta)_a$ at closed points of S, i.e., $(F_{\mathrm{irr}}^{\bullet}\mathcal{H}yp(\alpha,\beta))_a = F_{\mathrm{irr}}^{\bullet}(\mathcal{H}yp(\alpha,\beta))_a$. We have shown in Theorem 3.3.3 that the irregular Hodge filtration on the fibers $\mathcal{H}yp(\alpha,\beta)_a$ are given in terms of the cohomology classes $\omega_{r,s}$ in (3.3.3.2). Hence, we deduce that the irregular Hodge filtration on $\mathcal{H}yp(\alpha,\beta)$ is the one in assertion (1), and the irregular Hodge numbers are those given in (2). □

4 Frobenius structures on hypergeometric connections and p-adic estimates

In this section, let p be a prime number and $k = \mathbb{F}_q$ the finite field with $q = p^s$ elements for an integer $s \geq 1$. Let K be a finite extension of $\mathbb{Q}_p$ with residue field k containing an element π satisfying $\pi^{p-1} = -p$. We fix such an element π and denote the associated additive character by $\psi \colon \mathbb{F}_p \to K^{\times}$ [7, (1.3)]. The q-th power Frobenius on k admits a lift $\sigma = \mathrm{id}$ on $\mathcal{O}_K$.

Let $n > m$ be two integers, $\alpha = (\alpha_i = \frac{a_i}{q-1})_{i=1}^n, \beta = (\beta_j = \frac{b_j}{q-1})_{j=1}^m$ be two sequences of non-decreasing rational numbers $\in [0,1)$ with denominator $q-1$. Let $\omega : k^\times \to K^\times$ be the Teichmüller character and set $\chi_i = \omega^{a_i}$ (resp. $\rho_j = \omega^{b_j}$). The hypergeometric sum associated to $\psi, \chi = (\chi_1, \dots, \chi_n), \rho = (\rho_1, \dots, \rho_m)$ is defined, for $a \in k^\times$, by

$$\mathrm{Hyp}_{(n,m)}(\chi;\rho)(a) = \sum_{\substack{x_i, y_j \in k^\times, \\ x_1 \dots x_n = a y_1 \dots y_m}} \psi\left(\mathrm{Tr}_{k/\mathbb{F}_p}\left(\sum_{i=1}^n x_i - \sum_{j=1}^m y_j\right)\right) \cdot \prod_{i=1}^n \chi_i(x_i) \prod_{j=1}^m \rho_j^{-1}(y_j). \quad (4.0.0.1)$$

When (χ, ρ) is non-resonant, the above sum equals to (up to a sign) the Frobenius trace of the hypergeometric overconvergent F-isocrystal $\mathscr{H}yp(\chi,\rho)$ at $a \in \mathbb{G}_m(k)$ [27] and therefore can be written as a sum of n Frobenius eigenvalues. Its underlying connection is the hypergeometric connection $\mathcal{H}yp_{(-1)^{m+np}/\pi^{n-m}}$ [27, Thm. 4.1.3]. When (χ, ρ) is resonant, the above sum can be also written as a sum of n Frobenius eigenvalues (see § 4.2.1 for a direct proof by induction on n).

We are interested in the p-adic valuation of Frobenius eigenvalues (normalized by ord_q) of the above sum (called *Frobenius slopes*), encoded in the Frobenius Newton polygon [26, §2].

Recall that the irregular Hodge numbers of the hypergeometric connection $\mathcal{H}yp(\alpha;\beta)$ are given by the function $\theta\colon \{1,\dots,n\} \to \mathbb{Q}$ (1.0.1.1) is defined by

$$\theta(k) = (n-m)\alpha_k + \#\{i \mid \beta_i < \alpha_k\} + (n-k) - \sum_{i=1}^n \alpha_i + \sum_{j=1}^m \beta_j. \quad (4.0.0.2)$$

Definition 4.0.1. Let $\delta_1 < \dots < \delta_k$ be the Frobenius slopes of $\mathrm{Hyp}_{(n,m)}(\chi;\rho)(a)$, normalised by $\mathrm{ord}_q(q) = 1$, (resp. irregular Hodge numbers of $\mathcal{H}yp(\alpha,\beta)$) with multiplicity $\lambda_1 < \dots < \lambda_k$. The Newton polygon (resp. irregular Hodge polygon) is defined as the line in $\mathbb{R}^2$ joining P_i:

$$P_0 = (0,0), P_i = \left(\sum_{j=1}^i \lambda_j, \sum_{j=1}^i \lambda_j \delta_j\right), \quad i = 1, \dots, k.$$

Theorem 4.0.2. *Suppose $n > m$ and the orders of χ_i, ρ_j divide $p-1$. Then for each $a \in \mathbb{G}_m(k)$, the Frobenius Newton polygon of $\mathrm{Hyp}_{(n,m)}(\chi;\rho)(a)$ coincides with the irregular Hodge polygon defined by* (4.0.0.2).

A "Newton above Hodge" type result for twisted exponential sums was obtained by Adolphson and Sperber [3]. In our case, we show that the (combinatorial) Hodge polygon in *loc. cit.* for hypergeometric sums coincides with the irregular Hodge polygon of hypergeometric connections. Then, we apply a result of Wan [39] to conclude "Newton equals to Hodge". In [41], the second author and Zhu used a similar argument to study the Newton polygon of Kloosterman sums for classical groups.

4.1 Frobenius Newton polygon above Hodge polygon

In this subsection, we revise Adolphson–Sperber's definition of (combinatorial) Hodge polygon and their result on "Newton above Hodge" for certain twisted exponential sums [3]. Finally, we show that we can identity their Hodge polygon with the irregular Hodge polygon of hypergeometric connections (Proposition 4.1.7).

4.1.1. Let N be a positive integer,

$$\chi = (\chi_1, \dots, \chi_N) : (k^\times)^N \to K^\times$$

a multiplicative character, and $g : \mathbb{G}_m^N \to \mathbb{A}^1$ a morphism defined by a Laurent polynomial

$$g(x_1, \cdots, x_N) = \sum_{j=1}^{M} a_j x^{u_j} \in k[x_1^{\pm}, \cdots, x_N^{\pm}],$$

where $\{u_j\}_{j=1}^M$ is a finite subset of $\mathbb{Z}^N$ and $a_j \in k^\times$. For $m \in \mathbb{N}$, we consider the twisted exponential sum

$$S_m(\chi, g) = \sum_{x \in \mathbb{G}_m^N(\mathbb{F}_{q^m})} \chi^{(m)}(x)\psi^{(m)}(g(x)), \tag{4.1.1.1}$$

where $\chi^{(m)} = \chi \circ \mathrm{Nm}_{\mathbb{F}_{q^m}/k}$, $\psi^{(m)} = \psi \circ \mathrm{Tr}_{\mathbb{F}_{q^m}/\mathbb{F}_p}$. The associated L-function

$$L(\chi, g; T) = \exp\Big(\sum_{m \geq 1} S_m(\chi, g)\frac{T^m}{m}\Big) \tag{4.1.1.2}$$

is a rational function in T by the Grothendieck–Lefschetz trace formula (or the Dwork trace formula).

Recall that we denote $\Delta = \Delta(g)$ by the convex closure in $\mathbb{R}^N$ generated by the origin and lattices defined by the exponents $\{u_j\}$ of g in Definition 2.2.1. Let $C(g)$ be the cone over Δ, i.e., the union of all rays in $\mathbb{R}^N$ emanating from the origin and passing through Δ.

We set $M(g) = C(g) \cap \mathbb{Z}^N$. Adolphson and Sperber considered a subring $R(g)$ of $k[x_1^{\pm}, \ldots, x_N^{\pm}]$ defined by monomials with exponents in $M(g)$ [2, (1.7)]:

$$R(g) = k[x^{M(g)}].$$

We take $d_i \in [0, q-2]$ such that $\chi_i = \omega^{-d_i}$ [4]. We set

$$\overline{d_i} = \begin{cases} q - 1 - d_i & d_i \neq 0 \\ d_i & d_i = 0 \end{cases},$$

and

$$\mathbf{d} = (d_1, \ldots, d_N), \quad \overline{\mathbf{d}} = \{\overline{d_1}, \ldots, \overline{d_N}\}, \quad N_{\mathbf{d}} = (q-1)^{-1}\mathbf{d} + \mathbb{Z}^N.$$

We define a $R(g)$-module $R_{\mathbf{d}}(g)$ [2, (1.12)] by

$$R_{\mathbf{d}}(g) = \Big\{ \sum_{\text{finite}} b_u x^u | u \in N_{\mathbf{d}} \cap C(g), b_u \in k \Big\}.$$

There exists a (minimal) positive integer M such that for all $u \in \frac{\mathbb{Z}^N}{q-1} \cap C(g)$, the weight function $w(u)$, defined as the minimal positive real number w such that $u \in w\Delta(g)$, is a rational number with denominator dividing M. Then w defines an increasing filtration on $R(g)$ by

$$R(g)_{i/M} = \Big\{ \sum_{u \in M(g)} b_u x^u : w(u) \leq \frac{i}{M} \text{ for all } u \text{ with } b_u \neq 0 \Big\}.$$

We denote the associated graded module by

$$\overline{R}(g) = \bigoplus_{i \geq 0} \overline{R}(g)_{i/M}, \quad \overline{R}(g)_{i/M} = R(g)_{i/M}/R(g)_{(i-1)/M}.$$

Similarly, we equip $R_{\mathbf{d}}(g)$ with a filtration compatible with that of $R(g)$, and let $\overline{R}_{\mathbf{d}}(g)$ be the associated graded $\overline{R}(g)$-module.

[4] Adolphson–Sperber's convention $\chi_i = \omega^{-d_i}$ is different from our convention in the beginning of § 4 by a minus sign.

4.1.2. In the following, we assume that g is *non-degenerate* and that $\dim \Delta(g) = N$.

For $1 \leq i \leq N$, let $\overline{g}_i$ be the image of $x_i \frac{\partial}{\partial x_i} g$ in $\overline{R}(g)_1$, and set

$$\overline{I}_{\mathbf{d}} = \overline{g}_1 \overline{R}(g)_{\mathbf{d}} + \cdots + \overline{g}_N \overline{R}(g)_{\mathbf{d}}$$

a graded submodule of $\overline{R}(g)_{\mathbf{d}}$. For each $i \geq 0$, we define a finite set $\mathscr{B}^{i/M}_{\mathbf{d}} (= \mathscr{B}(g)^{i/M}_{\mathbf{d}}) \subset N_{\mathbf{d}} \cap C(g)$ of exponents as follows. We take a k-linearly independent set of monomials $\{x^\mu | \mu \in \mathscr{B}^{i/M}_{\mathbf{d}}\}$ of weight i/M which spans a k-subspace $\overline{V}_{\mathbf{d},i/M}$ complement to $\overline{R}(g)_{\mathbf{d},i/M} \cap \overline{I}_{\mathbf{d}}$, i.e.,

$$\overline{R}(g)_{\mathbf{d},i/M} = \overline{V}_{\mathbf{d},i/M} \bigoplus (\overline{R}(g)_{\mathbf{d},i/M} \cap \overline{I}_{\mathbf{d},i/M}).$$

We set $\mathscr{B}(g)_{\mathbf{d}} = \cup_{i \geq 0} \mathscr{B}(g)^{i/M}_{\mathbf{d}}$ and $V(g)$ the volume of $\Delta(g)$. The quotient $\overline{R}(g)_{\mathbf{d}}/\overline{I}_{\mathbf{d}}$ admits a basis of monomials in $S_{\mathbf{d}}$ and has dimension [3, Lem. 2.8]

$$\dim \overline{R}(g)_{\mathbf{d}}/\overline{I}_{\mathbf{d}} = N!V(g).$$

In this case, the L-function $L(\chi, g; T)^{(-1)^{N-1}}$ (4.1.1.2) is a polynomial of degree $N!V(g)$ [3, Cor. 2.12]. The q-order of roots of this polynomial are called *Frobenius slopes* of the twisted exponential sums $S_m(\chi, g)$.

Adolphson and Sperber studied the *Frobenius Newton polygon* defined by Frobenius slopes of this L-function (Definition 4.0.1) and compared it with a Hodge polygon defined as below.

For an integer $0 \leq d \leq q-2$, let d' be the nonnegative residue of pd modulo $q-1$. Recall that $q = p^s$ for an integer $s \geq 1$. For $\mathbf{d} = (d_1, \ldots, d_N)$, we set $\mathbf{d}' = (d'_1, \ldots, d'_N)$ and $\mathbf{d}^{(i)}$ the i-th composition of $(-)'$ on $\mathbf{d}$ for $i \geq 1$. Note that $\mathbf{d}^{(s)} = \mathbf{d}$.

We arrange elements of $S_{\mathbf{d}} = \{u_{\mathbf{d}}(1), \ldots, u_{\mathbf{d}}(N!V(g))\}$ by $w(u_{\mathbf{d}}(1)) \leq \cdots \leq w(u_{\mathbf{d}}(N!V(g)))$. And we repeat this ordering for $S_{\mathbf{d}'}, \ldots, S_{\mathbf{d}^{(s-1)}}$. For an integer $\ell \geq 0$, we set [3, Thm. 3.17]

$$W(\ell) = \mathrm{card}\left\{ j \;\middle|\; \sum_{i=0}^{s-1} w(u_{\mathbf{d}^{(i)}}(j)) = \frac{\ell}{M} \right\}.$$

When $\ell > sNM$, we have $W(\ell) = 0$.

Definition 4.1.3 (Adolphson–Sperber)**.** The Hodge polygon $\mathrm{HP}(\Delta(g)_{\mathbf{d}})$ is defined by the convex polygon in $\mathbb{R}^2$ with vertices $(0,0)$ and

$$\left(\sum_{\ell=0}^{m} W(\ell), \frac{1}{sM} \sum_{\ell=0}^{m} \ell W(\ell) \right), \quad m = 0, 1, \ldots, sNM.$$

Theorem 4.1.4 ([3, Cor. 3.18])**.** *If g is non-degenerate and $\dim(\Delta(g)) = N$, the Frobenius Newton polygon of $L(\chi, g; T)^{(-1)^{N-1}}$ lies above the Hodge polygon $\mathrm{HP}(\Delta(g)_{\mathbf{d}})$, and their endpoints coincide.*

Definition 4.1.5. We say that (g, χ) is *ordinary* if these two polygons coincide. When the character χ is trivial, we simply say g is *ordinary*.

4.1.6. In the following, we apply the above theory to the case of hypergeometric sums at the beginning of § 4. We may assume that χ_1 is trivial (i.e. $\alpha_1 = 0$). Let a be an element of $k^\times$. We take $N = n + m - 1$, $\mathbf{d} = (\overline{a_2}, \ldots, \overline{a_n}, b_1, \ldots, b_m)$, and g to be the non-degenerate function (2.2.1.1)

$$f_a = a \frac{y_1 \ldots y_m}{x_2 \ldots x_n} + x_2 + \cdots + x_n - y_1 - \cdots - y_m.$$

Then, we recover the hypergeometric sum (4.0.0.1) from (4.1.1.1).

Proposition 4.1.7. *If (χ,ρ) is non-resonant and the orders of the characters χ_i and ρ_j divide $p-1$, then the Hodge polygon* $\mathrm{HP}(\Delta(f_a)_{\mathbf{d}})$ *coincides with the irregular Hodge polygon defined by* (4.0.0.2) *associated to* $(0,\alpha_2=\frac{a_2}{p-1},\ldots,\alpha_n=\frac{a_n}{p-1})$, $(\beta_1=\frac{b_1}{p-1},\ldots,\beta_m=\frac{b_m}{p-1})$.

Proof. Since α_i,β_j have denominators dividing $p-1$, the numbers $\mathbf{d}^{(i)}$ are equal to $\mathbf{d}$ for every $i\geq 1$. In particular, the multi-set of slopes of $\mathrm{HP}(\Delta(f_a)_{\mathbf{d}})$ coincides with $w(S_{\mathbf{d}})=\{\omega(u)|u\in S_{\mathbf{d}}\}$.

The cohomology classes $\omega_{r,\ell}=g_{r,\ell}\cdot\eta$ in Proposition 3.2.1 form a basis of the de Rham cohomology group $\mathrm{H}^{n+m-1}_{\mathrm{dR}}(U_a,f_a)^{(G,\widetilde{\chi\times\rho})}$. By the calculation of cohomology groups [3, § 3, Thm. 3.14], the functions $\{g_{r,\ell}\}$ also form a basis of $\overline{V}_{\overline{\mathbf{d}}}$, with $\overline{\mathbf{d}}=(a_2,\ldots,a_n,\overline{b_1},\ldots,\overline{b_m})$. Hence

$$w(S_{\overline{\mathbf{d}}})=\{w(g_{r,\ell})|0\leq r\leq m,1\leq \ell\leq s_{r+1}-s_r\}.$$

By (3.1.0.4), Lemma 3.3.4 and the duality (3.3.6.4), the set of weights $w(S_{\mathbf{d}})$ coincides with the set of irregular Hodge numbers (4.0.0.2). Then, the proposition follows. □

4.2 Frobenius slopes of hypergeometric sums: proof of Theorem 4.0.2

We prove Theorem 4.0.2 by induction on n. Suppose the theorem holds when the rank of the hypergeometric F-isocrystal is less than n.

4.2.1. Resonant case. We first show that we can deduce the assertion in the resonant case from the induction hypothesis. We assume there exists i,j such that $\alpha_i=\beta_j$.

We slightly modify our convention on α,β by replacing those $\alpha_i,\beta_j=0$ by 1 and then arranging them as $0<\alpha_1\leq\alpha_2\leq\cdots\leq\alpha_n\leq 1$ and $0<\beta_1\leq\cdots\leq\beta_m\leq 1$. Note that this modification does not change the multi-set $\{\theta(1),\ldots,\theta(n)\}$ of irregular Hodge numbers. After twisting by a multiplicative character, we may assume that $\chi_n=\rho_m=1$ are the trivial characters (i.e., $\alpha_n=\beta_m=1$). Then we have the following identities:

$$\begin{aligned}
&\mathrm{Hyp}_{(n,m)}(\chi;\rho)(a) \qquad (4.2.1.1)\\
=\;&\sum_{x_i,y_j\in k^\times}\psi\Big(\sum_{i=1}^{n-1}x_i+a\frac{y_1\cdots y_m}{x_1\cdots x_{n-1}}-\sum_{j=1}^{m}y_j\Big)\cdot\prod_{i=1}^{n-1}\chi_i(x_i)\prod_{j=1}^{m-1}\rho_j^{-1}(y_j)\\
=\;&\sum_{x_i,y_j\in k^\times,y_m\in k}\psi\Big(\sum_{i=1}^{n-1}x_i-\sum_{j=1}^{m-1}y_j+y_m\Big(a\frac{y_1\cdots y_{m-1}}{x_1\cdots x_{n-1}}-1\Big)\Big)\cdot\prod_{i=1}^{n-1}\chi_i(x_i)\prod_{j=1}^{m-1}\rho_j^{-1}(y_j)\\
&-\sum_{x_i,y_j\in k^\times}\psi\Big(\sum_{i=1}^{n-1}x_i-\sum_{j=1}^{m-1}y_j\Big)\cdot\prod_{i=1}^{n-1}\chi_i(x_i)\prod_{j=1}^{m-1}\rho_j^{-1}(y_j)\\
=\;&q\,\mathrm{Hyp}_{(n-1,m-1)}(\chi';\rho')(a)-(-1)^{m-1}\prod_{i=1}^{n-1}G(\psi,\chi_i)\prod_{j=1}^{m-1}G(\psi,\rho_j^{-1}),
\end{aligned}$$

where $\chi'=(\chi_1,\ldots,\chi_{n-1})$, $\rho'=(\rho_1,\ldots,\rho_{m-1})$, and $G(\psi,\chi_i)=\sum_{x\in k^\times}\psi(x)\chi_i(x)$ denotes the Gauss sum. In particular, the above sum can be written as a sum of n Frobenius eigenvalues by induction. Let θ' be the function (4.0.0.2) defined by rational numbers $\alpha_1,\ldots,\alpha_{n-1},\beta_1,\ldots,\beta_{m-1}$. Then, we have

$$\theta(k)=\theta'(k)+1,\ \forall\ 1\leq k\leq n-1$$

and

$$\theta(n)=\sum_{i=1}^{n}(1-\alpha_i)+\sum_{\beta_j<1}\beta_j=\mathrm{ord}_q\Big(\prod_{i=1}^{n-1}G(\psi,\chi_i)\prod_{j=1}^{m-1}G(\psi,\rho_j^{-1})\Big),$$

where the second identity follows from Stickelberger's theorem, saying that

$$\operatorname{ord}_q G(\psi, \omega^{-k}) = \frac{k}{p-1}.$$

Then, the theorem in the resonant case follows from the induction hypothesis and decomposition (4.2.1.1).

4.2.2. Non-resonant case. By the previous argument, we may assume that the assertion for the hypergeometric sum of type (n, m) defined by a resonant pair (α, β) is already proved. It suffices to treat the non-resonant case. We may assume $\chi_1 = 1$ is trivial.

We set $\widetilde{f}_a(x_2, \dots, x_n, y_1, \dots, y_m) = f_a(x_2^{p-1}, \dots, x_n^{p-1}, y_1^{p-1}, \dots, y_m^{p-1})$. We first prove the ordinariness of exponential sums associated to $\widetilde{f}_a$ (Definition 4.1.5) using a theorem of Wan [39].

Let $\delta_1, \cdots, \delta_{m+n}$ be all the facets of $\Delta = \Delta(\widetilde{f}_a)$ which do not contain the origin. Let $\widetilde{f}_a^{\delta_i}$ be the restriction of $\widetilde{f}_a$ to δ_i [40, §1.1], which is also non-degenerate [40, §3.1]. By [40, Thm. 3.1], $\widetilde{f}_a$ is ordinary if and only if each $\widetilde{f}_a^{\delta_i}$ is ordinary.

Each Laurent polynomial $\widetilde{f}_a^{\delta_i}$ is diagonal, that is, $\widetilde{f}_a^{\delta_i}$ has exactly $n+m-1$ non-constant terms of monomials and $\Delta(\widetilde{f}_a^{\delta_i})$ is $(n+m-1)$-dimensional [40, § 2]. Indeed, if $V_1, \cdots, V_{m+n-1}$ denote the vertex of δ_i written as column vectors, the set $S(\delta_i)$ of solutions of

$$(V_1, \cdots, V_{m+n-1}) \begin{pmatrix} r_1 \\ \vdots \\ r_{m+n-1} \end{pmatrix} \equiv 0 \pmod 1, \quad r_i \text{ rational}, \ 0 \le r_i < 1,$$

forms an abelian group, which is isomorphic to $(\mathbb{Z}/(p-1)\mathbb{Z})^{n+m-1}$. We deduce that for each δ_i, $\widetilde{f}_a^{\delta_i}$ is ordinary by [40, Cor. 2.6].

We have a decomposition of exponential sums as follows:

$$\sum_{x_i, y_j \in k^\times} \psi(\widetilde{f}_a(x_i, y_j)) = \sum_{\chi_i, \rho_j} \mathrm{Hyp}_{(n,m)}(\chi, \rho)(a), \tag{4.2.2.1}$$

where the sum is taken over all multiplicative characters χ_i, ρ_j with $2 \le i \le n, 1 \le j \le m$ of orders dividing $p-1$. We have a similar decomposition for $\mathscr{B}_{\mathbf{d}}$ (§ 4.1.2) given by

$$\mathscr{B}_1(\widetilde{f}_a) = \bigsqcup_{\mathbf{d}} \mathscr{B}_{\mathbf{d}}(f_a),$$

where $1 = (0, 0, \dots, 0)$ and $\mathbf{d}$ is taken over all $(n+m-1)$-tuple of rational numbers with denominators $p-1$ in $[0, 1)$.

On the left-hand side of (4.2.2.1), we have shown "Newton equals to Hodge" (i.e. the ordinariness of $\widetilde{f}_a$). Together with the "Newton above Hodge" for each hypergeometric sum (Theorem 4.1.4), we deduce that "Newton equals to Hodge" for each component of the right-hand side. Then, the assertion in the non-resonant case follows from Proposition 4.1.7. □

In particular, our proof shows Proposition 4.1.7 in the resonant case.

Corollary 4.2.3. *Proposition 4.1.7 holds without the non-resonant assumption.*

Proof. In the resonant case, the Frobenius Newton polygon equals the irregular Hodge polygon by §4.2.1. By the proof in §4.2.2, the Frobenius Newton polygon equals the (combinatorial) Hodge polygon defined by Adolphson–Sperber. Then, the assertion follows. □

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